



Inner and outer approximation of the polar diagram of a sailboat

Pau Herrero*, Luc Jaulin**, Josep Vehí* and Miguel A. Sainz*

(*)Institut d'Informàtica i Aplicacions (IliA)

Universitat de Girona

Girona - Catalonia - Spain

()Extraction et Exploitation de l'Information en Environnements Incertains (E3I2)**

ENSIETA

Brest - France



- Problem Statement
- Example: Polar Diagram of a Sailboat
- Quantified Set Inversion Algorithm
- Sailboat Control Application



- Many control theory problems have a solution set that can be written into the form

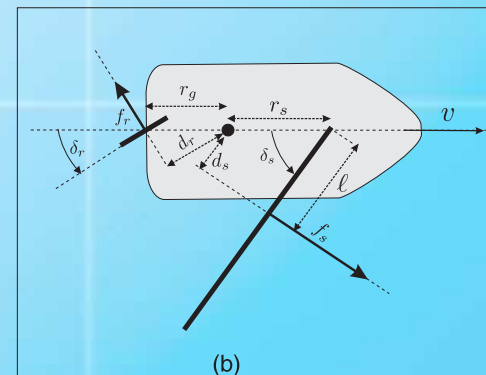
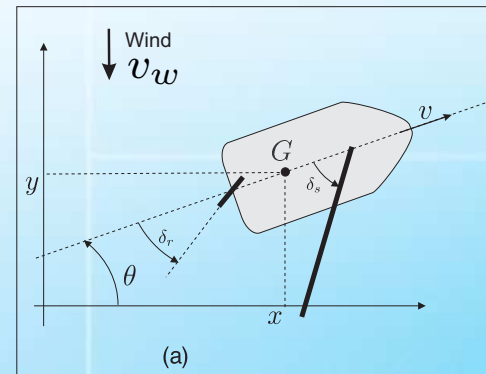
$$\mathcal{S} \triangleq \{ \mathbf{p} \in \mathbf{P} \mid (\exists \mathbf{q} \in \mathbf{Q}) \mathbf{f}(\mathbf{p}, \mathbf{q}) = \mathbf{0} \},$$

- where \mathbf{P} is a box of \mathbb{R}^{n_p} , \mathbf{Q} is a box of \mathbb{R}^{n_q} and \mathbf{f} is a continuous function from $\mathbb{R}^{n_p+n_q}$ to \mathbb{R}^{n_f} .
- **Challenging problem:** General algorithm able to find an **inner** and an outer approximation of \mathcal{S} .

Example: Sailboat Modelization



$$\left\{ \begin{array}{l} \dot{x} = v \cos \theta, \\ \dot{y} = v \sin \theta - \beta v_w, \\ \dot{\theta} = \omega, \\ \dot{\delta}_s = u_1, \\ \dot{\delta}_r = u_2, \\ \dot{v} = \frac{f_s \sin \delta_s - f_r \sin \delta_r - \alpha_f v}{m}, \\ \dot{\omega} = \frac{(\ell - r_s \cos \delta_s) f_s - r_r \cos \delta_r f_r - \alpha_\theta \omega}{J}, \\ f_s = \alpha_s (v_w \cos (\theta + \delta_s) - v \sin \delta_s), \\ f_r = \alpha_r v \sin \delta_r. \end{array} \right.$$



- The state vector $\mathbf{x} = (x, y, \theta, \delta_s, \delta_r, v, \omega)^T \in \mathbb{R}^7$. The inputs u_1 and u_2 of the system are the derivatives of the angles δ_s and δ_r .

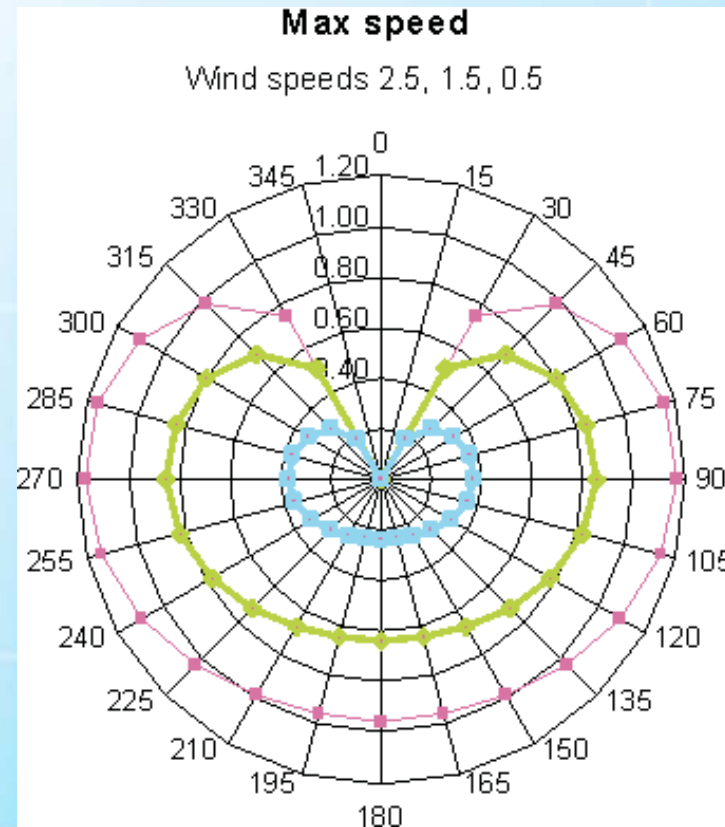
Example: Polar Diagram of a Sailboat

MiceLab
Model Interact & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

- The *polar diagram* of the sailboat is the set \mathcal{S} of all pairs (θ, v) that can be reached by the boat, in a cruising mode.



Example: Polar Diagram of a Sailboat

MiceLab
Model Interact & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

- In boat's **cruising mode**, the speed of the boat, its angular velocity, . . . are assumed to be constant, i.e.,

$$\dot{\theta} = 0, \dot{\delta}_s = 0, \dot{\delta}_r = 0, \dot{v} = 0, \dot{\omega} = 0.$$

Then, we get

$$\left\{ \begin{array}{l} 0 = \frac{f_s \sin \delta_s - f_r \sin \delta_r - \alpha_f v}{m}, \\ 0 = \frac{(\ell - r_s \cos \delta_s) f_s - r_r \cos \delta_r f_r}{J}, \\ f_s = \alpha_s (v_w \cos (\theta + \delta_s) - v \sin \delta_s), \\ f_r = \alpha_r v \sin \delta_r. \end{array} \right.$$

Example: Polar Diagram of a Sailboat

MiceLab
Model, Integrate & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

- The polar diagram can be written as

$$\mathbb{S} = \{(v, \theta) \mid \exists \delta_r, \exists \delta_s, \mathbf{f}(v, \theta, \delta_r, \delta_s) = \mathbf{0}\},$$

where

$$\mathbf{f}(v, \theta, \delta_r, \delta_s) = \begin{pmatrix} \alpha_s (v_w \cos(\theta + \delta_s) - v \sin \delta_s) \sin \delta_s - \alpha_r v \sin^2 \delta_r - \alpha_f v \\ (\ell - r_s \cos \delta_s) \alpha_s (v_w \cos(\theta + \delta_s) - v \sin \delta_s) - r_r \alpha_r v \sin \delta_r \cos \delta_r \end{pmatrix}$$

Example: Polar Diagram of a Sailboat

MiceLab
Model Interact & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

- An elimination of δ_r yields to

$$\mathbb{S} = \{(\theta, v) | (\exists \delta_s \in [-\frac{\pi}{2}, \frac{\pi}{2}]) f(\theta, v, \delta_s) = 0\},$$

where

$$f(\theta, v, \delta_s) = ((\alpha_r + 2\alpha_f)v - 2\alpha_s v_w \cos(\theta + \delta_s) \sin \delta_s + 2\alpha_s v \sin^2 \delta_s)^2 + \left(\frac{2\alpha_s}{r_r} (\ell - r_s \cos \delta_s) (v_w \cos(\theta + \delta_s) - v \sin \delta_s)\right)^2 - \alpha_r^2 v^2.$$

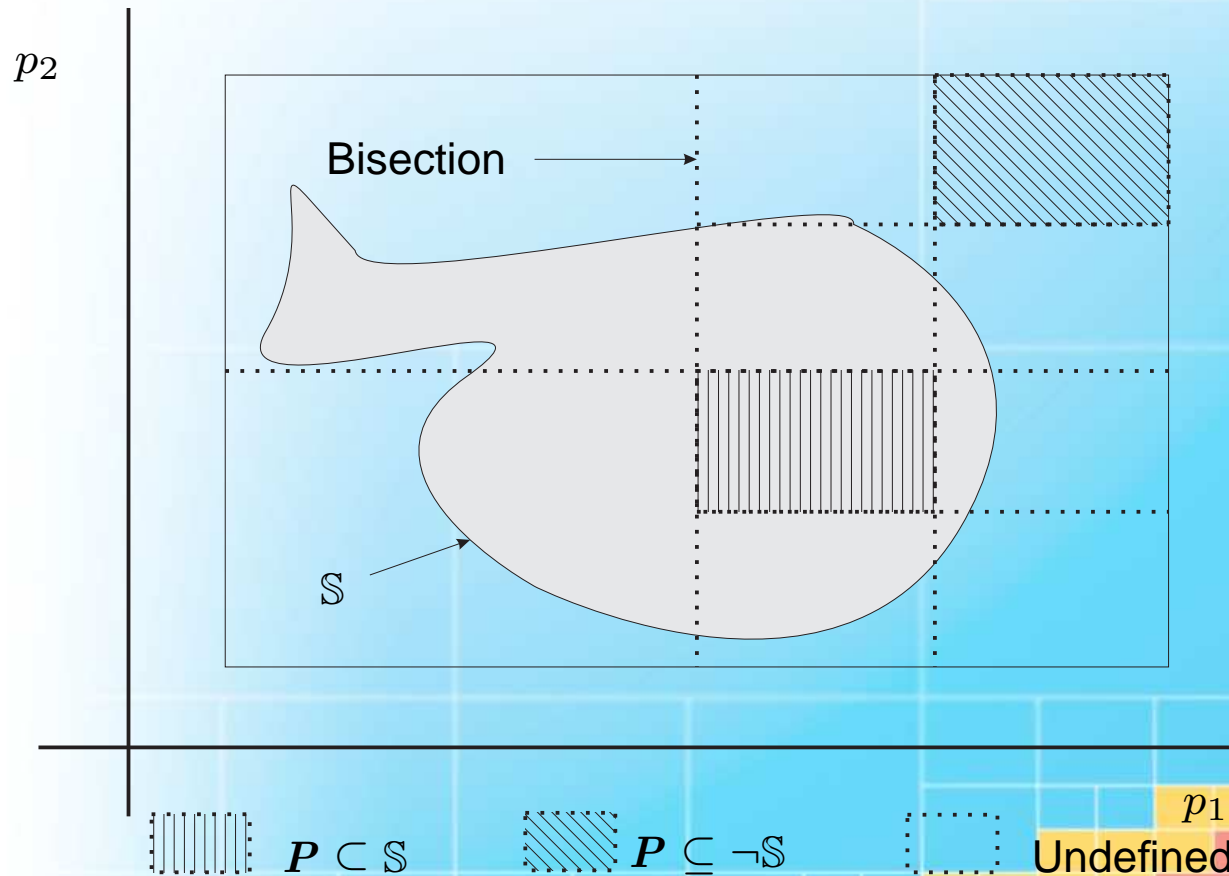
Quantified Set Inversion Algorithm

MiceLab
Model Inverse & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

$$\mathcal{S} \triangleq \{p \in P \mid (\exists q \in Q) f(p, q) = 0\},$$





$$\mathcal{S} \triangleq \{p \in P \mid (\exists q \in Q) f(p, q) = 0\},$$

Rule 1 : $P \subseteq \mathcal{S} \Leftrightarrow (\forall p \in P)(\exists q \in Q) f(p, q) = 0,$

Rule 2 : $P \subseteq \neg\mathcal{S} \Leftrightarrow (\forall p \in P)\neg((\exists q \in Q) f(p, q) = 0),$

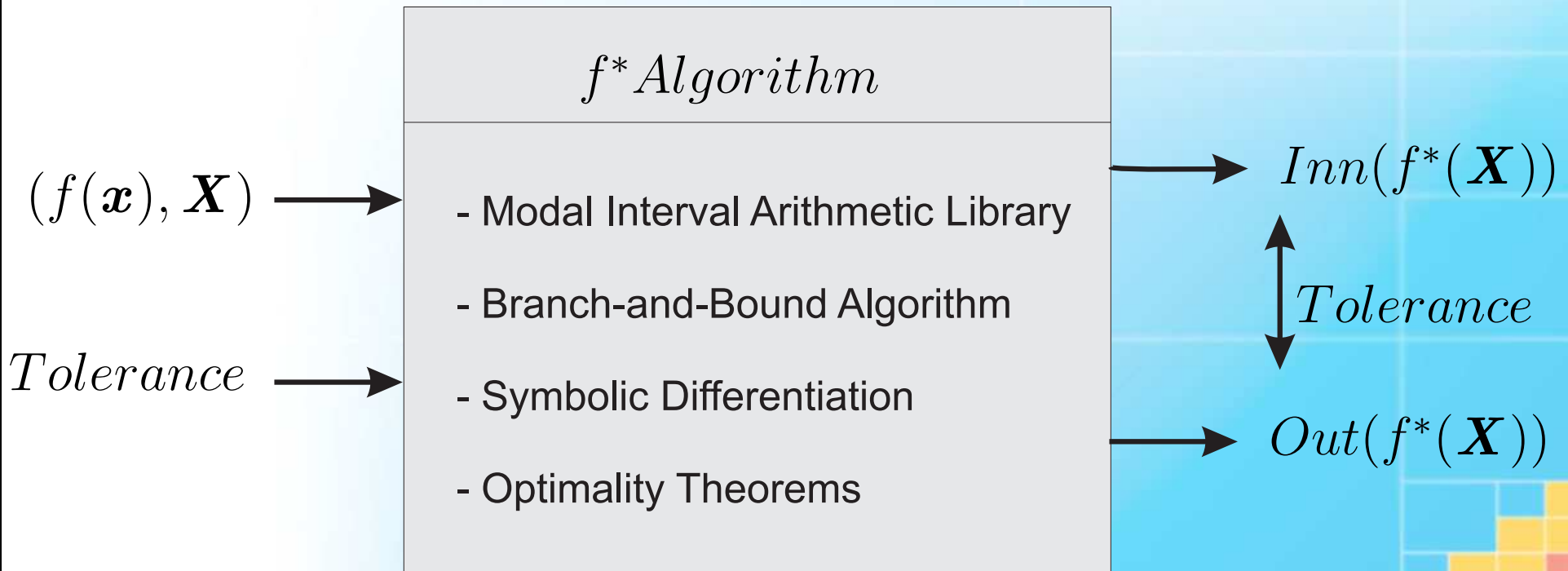
Rule 3 : *Undefined* $\Leftrightarrow \neg(\text{Rule 1}) \wedge \neg(\text{Rule 2}).$



- **MIA** makes equivalent a $\forall\exists$ -first order constraint to an interval inclusion.

$$\textit{Rule 1} : (\forall p \in P)(\exists q \in Q)f(p, q) = 0 \iff f^*(P, Q) \subseteq [0, 0].$$

$$\textit{Rule 2} : (\forall p \in P)\neg((\exists q \in Q)f(p, q) = 0) \iff f^*(P, Q) \not\subseteq [0, 0].$$





- Check if $(\Theta, V) = ([\pi, 2\pi], [2.5, 5]) \subseteq \mathbb{S}$

$$\begin{aligned} & (\forall \theta \in [\pi, 2\pi])(\forall v \in [2.5, 5])(\exists \delta_s \in [-\pi/2, \pi/2]) \\ & \left((\alpha_r + 2\alpha_f)v - 2\alpha_s v_w \cos(\theta + \delta_s) \sin \delta_s + 2\alpha_s v \sin^2 \delta_s \right)^2 \\ & + \left(\frac{2\alpha_s}{r_r} (l - r_s \cos \delta_s) (v_w \cos(\theta + \delta_s) - v \sin \delta_s) \right)^2 - \alpha_r^2 v^2 = 0. \end{aligned}$$

- where $v_w = 10$, $\alpha_f = 60$, $\alpha_r = 300$, $r_s = 1$, $r_r = 2$, $l = 1$.
- Since,

$$\text{Out}(fR^*(\theta, v, \delta_s)) = [1.1e + 05, -9.3e + 03] \subseteq [0, 0] \Rightarrow (\Theta, V) \in \mathbb{S}.$$

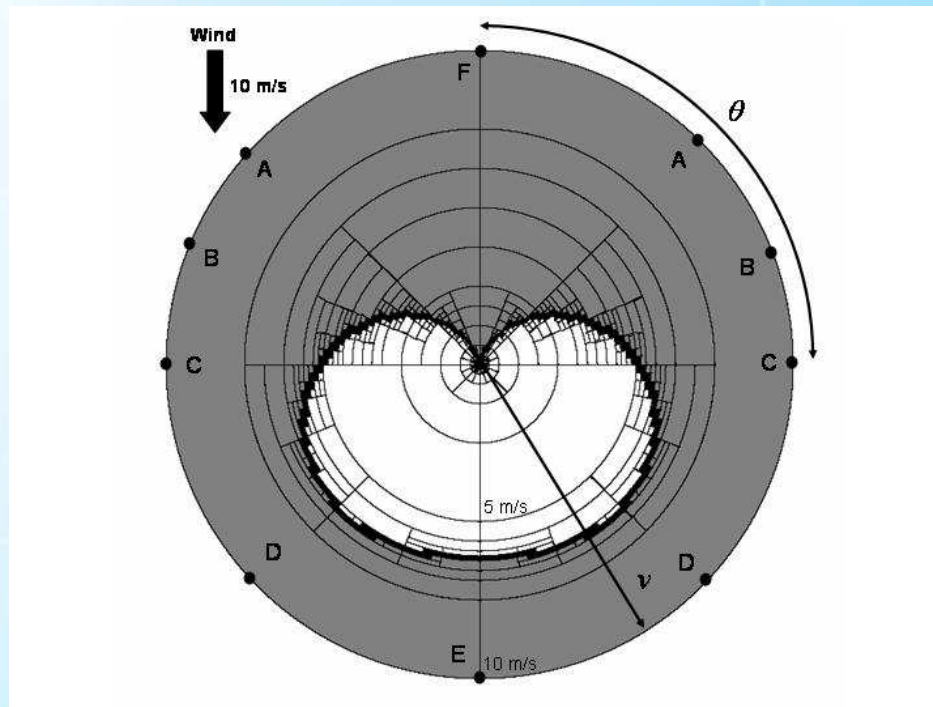
Polar Diagram - Test Case

MiceLab
Model Interact & Control Engineering



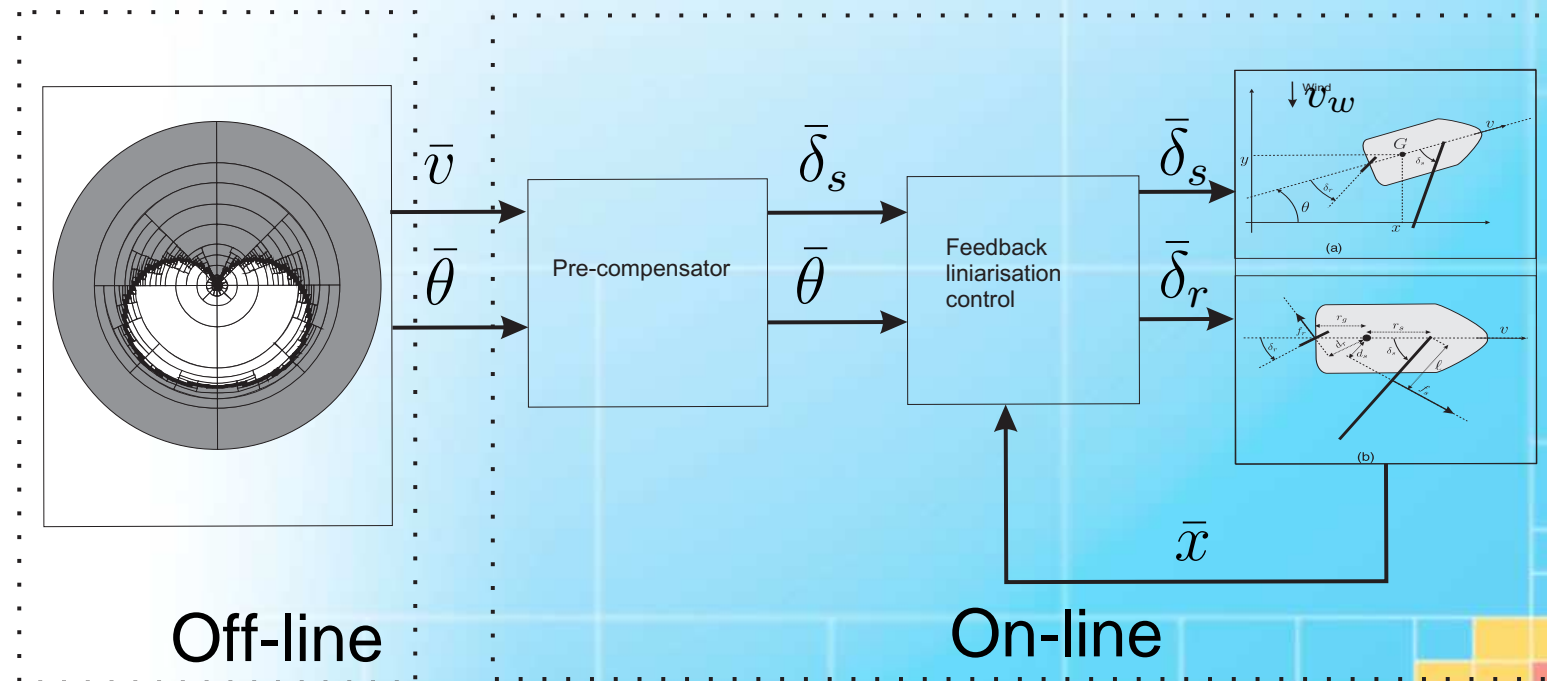
ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

- For $v_w = 10$, $\alpha_f = 60$, $\alpha_r = 300$, $r_s = 1$, $r_r = 2$, $l = 1$, and a search space (Θ, V) of $([0, 2\pi], [0, 20])$, the following solution is obtained.





- **Objective:** Automatically control the speed v and the orientation θ of a sailboat.



Control Application

MiceLab
Model Interact & Control Engineering



ENSIETA
ÉCOLE NATIONALE SUPÉRIEURE D'INGÉNIEURS

