Power Spectra of Random Spikes and Related Complex Signals

with Application to Communications

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Abstract

"Random spikes" belong to the common language used by engineers, physicists and biologists to describe events associated with time records, locations in space, or more generally, space-time events. Indeed, data and signals consisting of, or structured by, sequences of events are omnipresent in communications, biology, computer science and signal processing. Relevant examples can be found in traffic intensity and neurobiological data, pulse-coded transmission, and sampling.

This thesis is concerned by random spike fields and by the complex signals described as the result of various operations on the basic event stream or spike field, such as filtering, jittering, delaying, thinning, clustering, sampling and modulating. More precisely, complex signals are obtained in a modular way by adding specific features to a basic model. This modular approach greatly simplifies the computations and allows to treat highly complex model such as the ones occurring in ultra-wide bandwidth or multipath transmissions.

We present a systematic study of the properties of random spikes and related complex signals. More specifically, we focus on second order properties, which are conveniently represented by the spectrum of the signal. These properties are particularly attractive and play an important role in signal analysis. Indeed, they are relatively accessible and yet they provide important informations.

Our first contribution is theoretical. As well as presenting a modular approach for the construction of complex signals, we derive formulas for the computation of the spectrum that preserve such modularity: each additional feature added to a basic model appear as a separate and explicit contribution in the corresponding basic spectrum. Moreover, these formula are very general. For instance, the basic point process is not assumed to be a homogeneous Poisson process but it can be any second order stationary process with a given spectrum. In summary, they provide very useful tools for model analysis.

We then give applications of the theoretical results: spectral formulas for traffic analysis, pulse based signals used in spread spectrum communications, and randomly sampled signal.

Résumé

Les impulsions aléatoires, ou processus ponctuels, sont couramment employés par les ingénieurs, physiciens ou biologistes pour décrire des événements associés à des données temporelles, des positions spatiales, ou, plus généralement, à des événements spatio-temporels.

Les données et signaux composés ou structurés par des suites d'événements sont en effet omniprésents en communications, en biologie, dans les sciences informatiques et en traitement du signal. On en trouve de nombreux exemples dans les reseaux de communications, les données neurobiologiques, les transmissions par codage à impulsions et l'échantillonnage.

Cette thèse traite des processus ponctuels et des signaux complexes décrits comme le résultat d'operations variées sur une séquence d'événements de base, telles que le filtrage, les déplacements et les pertes aléatoires des points, le "clustering", l'échantillonage et la modulation. Plus précisement, on obtient des signaux complexes d'une façon modulaire en ajoutant des propriétés spécifiques à un modèle de base. Cette approche modulaire simplifie considérablement les calculs et permet de traiter des modèles complexes comme ceux apparaissant dans les communications à large bande en présence de réflections multiples.

Nous présentons une étude systématique des propriétés des champs d'impulsions alèatoires et des signaux complexes associés. Plus spécifiquement, nous nous concentrons sur les propriétés du second ordre représentées par le spectre du signal.

Notre première contribution est théorique. Parallèlement à la présentation d'une approche modulaire pour la construction des signaux complexes, nous dérivons des formules pour le calcul du spectre qui préservent cette modularité : chaque propriété additionnelle ajoutée au modèle de base apparait comme une contribution séparée et explicite dans le spectre de base correspondant. De plus, ces formules sont très générales. Par exemple, on ne suppose pas que le processus de base est un processus de Poisson homogène : il peut être n'importe quel processus stationnaire du second ordre avec un spectre donné.

Nous donnons des exemples d'application de ces résultats théoriques à travers des formules spectrales pour l'analyse du trafic dans les reseaux, des signaux à modulation par impulsions utilisés dans les communications à large bande ou encore des signaux échantillonnés aléatoirement.

Acknowledgements

Acknowledgements are definitively the most fun part of any Ph.D. thesis: they are fun to write, and, usually, even fun to read (at least for the author). They are also the most read part since "inspected" by all the colleagues checking if their name is mentioned.

With an absolute lack of originality, I start these acknowledgments by expressing my deepest gratitude to my Ph.D advisor, Pierre Brémaud. As well as being a unique advisor ("jouons le violon" ...), he is, most of all, a master of an outstanding humor and sarcasm. I have deeply enjoyed our "once every two weeks" discussions, the most interesting ones taking place in some Lausanne restaurant chosen on a minimum distance criterion. I think he had in mind a "giro d'Italia" or "tour de France" scenario: he would have led me for a while, and then let me go alone to the finish line. Well, I have ridiculously crashed with my bike several times, and, as many have remarked, broken several bones. I got to the finish line while he was comfortably watching the tour, eating "carambars" and "maltesers". Luckily, he had left me some.

Martin Vetterli has co-directed my thesis and coached my staying at LCAV since I have knocked on his door as a refugee escaping from rainy Paris. He has already received gazillions of acknowledgements for his outstanding mentoring, and with this respect, it is hard to add more to what my predecessors have already said. However, I want to express my deepest gratitude for sharing with me his views on research and teaching and for all the more or less personal discussions we had. Although short, they were very intense and franc. Moreover, they were held in extremely fascinating places, like mountain trail paths, trains, and train stations. I hope not to have occupied his red-sofa too much.

Laurent Massoulié has considerably contributed to the fundamentals of my thesis. I thank him very much, hoping that next time our collaboration will be longer and not just e-mail based.

Jean-Yves Le Boudec, Emre Telatar, David Vere-Jones, and Moe Win kindly accepted to be members of my thesis jury. I hope they have enjoyed reading my manuscript, even if no funny nor philosophical citations appeared at the beginning of each chapter. Emre Telatar has also provided the "soundtrack" of my thesis writing, by letting me discover the beauty of the six Bach suites for solo cello, performed by M. Rostropovich. Moe Win gave me the opportunity to spend a few months with his group at MIT, and our fruitful discussions led to some interesting UWB results presented in this thesis. David Vere-Jones shared with me the pleasure of walking in the heights of Lausanne, and the bad luck of a week of stratus that prevented us from enjoying the view on the lake and the mountains.

I am very thankful to LCAV members and ex-members, and in particular to the "clique" created in the "PSE satellite": I have shared my office and my mess with "le petit Thibaut", some snow slopes with Guillermo Barextezxecxaxe and some geeky talks with Henri. Special thanks to Don Julius (the overseas LCAV member): as well as being the "king of international connections", he introduced me to the world of biking and bike crashing. In his short internship at LCAV, Justin has tracked all the mistakes I have made in the first version of this thesis and he has revealed the true fact that I am not able to make computations.

A gazillions of thanks to all the people that made my Ph.D. period extremely enjoyable: LTHI, LTHC, LCM, and MICS members; mountaineering friends, for making lots of my week-ends much more interesting, keeping me away from too much of insane working on Saturdays and Sundays (actually, not an hard task to do); Yosemite staff, for always warmly welcoming my almost daily visits; Bobby Sassi, for weekly breaking the promise to quit academia as soon as possible; Angela, Lamberto e Fabrizio, for not worrying too much if, although I grew up, I still want to be a fireman, a ballet dancer or a spaceship commander (may the force be with you).

* * *

I have decided not to put any philosophical citations at the beginning of each chapter of my thesis (let pretend that it is a form of protest to philosophers who do not put mathematical formulas at the beginning of their masterpieces). However, there are a few citations I would love to share: I found them funny and I hope they will make you laugh or, at least, rise a smile. Moreover, I think they are the most pertinent ones with this thesis.

From the moment I picked up your book until I laid it down, I was convulsed with laughter. Some day I intend reading it.

Groucho Marx

Scientifiquement, c'est pas mieux, comme vous me voyez, je suis incapable de reconnaître un rayon laser d'une corde à linge ordinaire, ou un chien qui pète d'un avion qui renifle. Le savant le savait bien, lui, que sans la science l'homme ne serait qu'un stupide animal sottement occupé à s'adonner aux vains plaisirs de l'amour dans les folles prairies de l'insousciance, alors que la science, et la science seule, a su lui apporter patiemment, au fil des siècles, le parcmètre automatique et l'horloge pointeuse sans lesquels il n'est pas de bonheur terrestre possible. C'est quand même grâce aux progrès fantastiques de la science que désormais nous savons que, quand on plonge un corps dans une baignoire, le téléphone sonne. C'est grâce aux progrès fantastiques de la science que désormais l'homme peut se rendre, en moins de trois heures, de Moscou à Varsovie. Et si y avait pas la science, si y avait pas la science, malheureux colportes, boursouflés d'ingratitude aveugle et d'ignorance crasse, si y avait pas la science, combien d'entre nous pourraient profiter de leur cancer pendant plus de cinq ans ? Et n'est-ce pas pas le triomphe absolu de la science que d'avoir permis aujourd'hui, sur la seule décision d'un vieillard californien impuissant, ou d'un fossile ukrainien encore plus gâteux que l'autre, l'homme puisse en une seconde faire sauter quarante fois sa planète, sans bouger les oreilles ! C'est pas moi qui le dis, c'est Fucius, croyez-moi, il avait oublié d'être con. Fucius disait : "Une civilisation sans la science, c'est aussi absurde qu'un poisson sans bicyclette.

Pierre Desproges

[...] *presto ti accorgerai com'è facile farsi un'inutile software di scienza, e vedrai che confuso problema è adoprare la propria esperienza ...*

Francesco Guccini

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This document, including the color cover page, has been written in **LaTeX** with **PSTricks** macros to generate most of the figures. It has been edited using **Xemacs** running under **Linux Debian**. I am deeply grateful to all the people that contribute the richness of the **Open Source** world, because it is not just a matter of free software, but of free ideas and free choice.

Contents

Introduction

Motivation

"Random spikes" belong to the common language used by engineers, physicists and biologists to describe events associated with time records, locations in space, or more generally, space-time events. Indeed, data and signals consisting of, or structured by, sequences of events are omnipresent in communications, biology, computer science and signal processing. Relevant examples can be found in traffic intensity and neurobiological data, pulse-coded transmission, and sampling. For instance, the exchange of information sometimes takes place by mean of pulse sequences, as in communication systems, where different type of pulse modulations are designed to encode the information, or in biology, where neurons naturally communicate by firing spikes. In a spatial setup, spikes can model a configuration of points or positions: mobile units in mobile communications systems; nodes in a network; etc.

In one dimension, sequences of spikes are commonly called *random spike trains*, while in the spatial case, they are referred to as *random spike fields*. In mathematical parlance, they are point processes.

We are concerned by random spike fields and by the complex signals described as the result of various operations on the basic event stream or spike field: filtering, jittering, delaying, thinning, clustering, sampling and modulating. More specifically, we shall consider three classes of signals:

(i) the random *spike fields* themselves;

(ii) the *filtered* random spike fields;

(iii) the *modulated* random spike fields.

They represent the basic models. Complex signals that are of interest in many domains of application, such as the ones mentioned above, can be then obtained by adding specific features to these basic models in a modular way. To illustrate this with an example, we consider a pulse coded modulation: Starting from a basic temporal structure given by the system clock and modelled as a T-regularly spaced spike train (Figure 1(a)), the information to be transmitted is coded as relative displacements with respect to the regular grid (Figure 1(b)); Jitter is then introduced as irregularities of the system clock that perturbs the temporal structure (Figure 1(c)); The physical pulse shaped transmitted signal is obtained by a filtering operation (Figure 1(d)) where the pulses can be modulated in view of adding more information (Figure 1(e)); Interferences in the transmitting device may causes losses of the pulses (Figure 1(f)).

This model is considered later in the main text, with additional features of practical interest that can be added one by one, modularly. This systematic approach greatly simplifies the computations and allows to treat highly complex models such as the ones occurring in Ultra-Wide Bandwidth (UWB) or multipath transmissions.

This thesis presents a detailed study of the properties of random spikes and related complex signals. More precisely, it focuses on second order properties, which are conveniently repre-

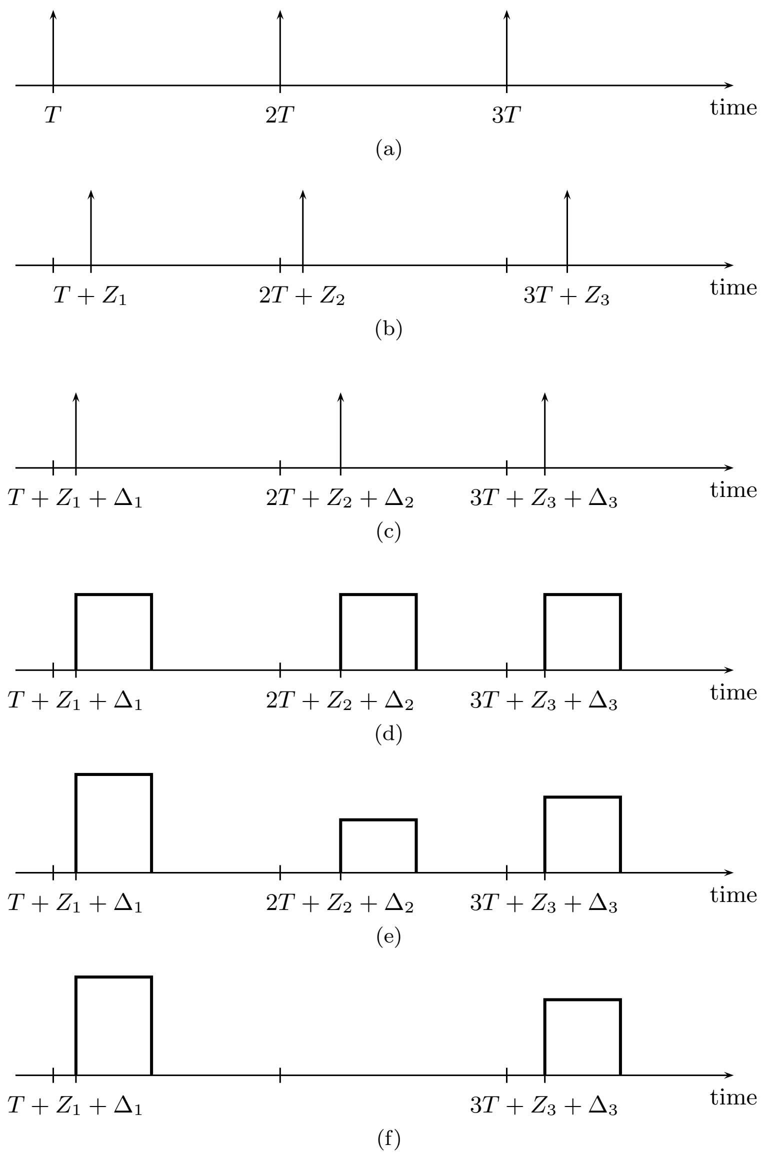


Figure 1: Modular approach to complex signals: (a) Basic event stream; (b) Jittering (information supported by the delays); (c) Jittering (clock jitter); (d) Filtering (pulse shaping); (e) Modulation (information supported by the amplitudes); (f) Thinning (losses).

sented by the power spectrum of the signal. These properties are particularly attractive and play an important role in signal analysis, since they are relatively accessible, and yet they provide important informations. In communication systems, for instance, the power spectrum gives informations on the frequency band occupation of transmitted signals. Such informations are crucial for transmitting system design since band occupation is submitted to regulatory constraints. Figure 2 shows the Federal Communication Committee (FCC) regulatory mask for UWB indoor transmissions (`www.fcc.gov`).

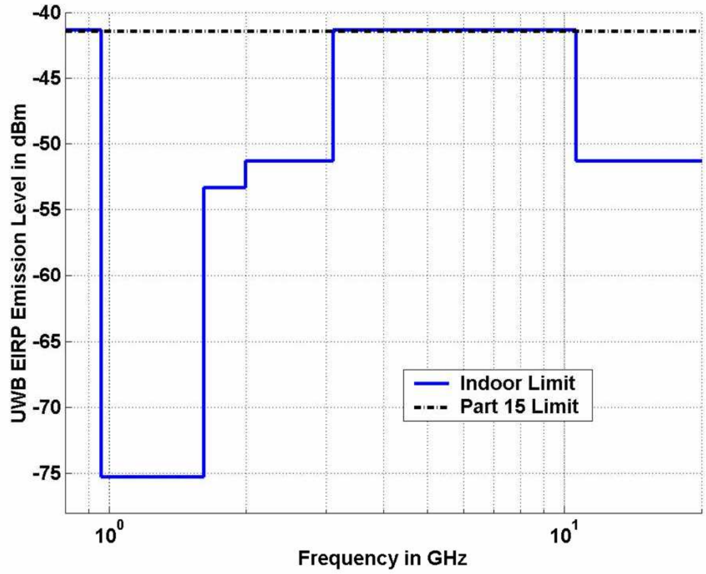


Figure 2: Federal Communication Committee regulatory mask for UWB indoor transmissions. The mask imposes an upper limit to the spectral energy of UWB signals in order to avoid interferences with critical narrow band systems, such as GPS.

Our first contribution is theoretical. As well as presenting a modular approach for the construction of complex signals, we derive formulas for the computation of the spectrum that preserves such modularity: each additional feature added to a basic model appears as a separate and explicit contribution in the corresponding basic spectrum. Moreover, these formula are very general. For instance, the basic point process is not assumed to be a homogeneous Poisson process but it can be any second order stationary process with a Bartlett spectrum (this notion will be explained in the main text). In summary, they provide very useful tools for model design and analysis.

We then give applications of the theoretical results: spectral formulas for traffic analysis, pulse based signals used in spread spectrum communications, and randomly sampled signal.

Related Work and Original Contributions

Random Spike Fields

Random spike fields represent the basic processes on which are built, through various operation, the complex processes considered in this work. Mathematically, they are aptly represented by point processes.

State of the Art. Earlier reviews of the spectral theory of random spike fields were presented by Daley [1970]; Neveu [1977]; Brillinger [1981]; Daley and Vere-Jones [1988, 2002]. In particular, the works of Neveu [1977] and Daley and Vere-Jones [1988, 2002] provide existence and uniqueness theorems for the Bartlett spectral measure of a point processes.

We also consider a particular class of spike fields, the *Hawkes branching point processes.* Hawkes processes were introduced, under the name of self-exciting point processes, by Hawkes [1971], and further studied in [Hawkes, 1974]; see also [Daley and Vere-Jones, 1988, 2002]. Such branching point processes are of interest in epidemics, and also in seismology, where they are known as ETAS models (see [Ogata, 1988]). They have also been used to model neuronal activity in the brain (see [Johnson, 1996]).

Earlier results in the computation of the power spectrum of Hawkes process appear in [Brémaud and Massoulié, 2002], where however the study is restricted to the line. See also [Brémaud and Massoulié, 2001], where the critical case, leading to long-range dependence, was considered.

Original Contribution. We compute the power spectrum of classic spike fields. Our contribution is to provide, with more precision than the general theory, the characteristics of the test functions appearing in the definition of the Bartlett power spectrum. Indeed the usual general conditions such as compactness of support, or rapid decay are not satisfactory for many purposes. In turn, such a detailed analysis allows to determine the condition of validity of the power spectrum of complex signals, such as the filtered and modulated spike fields.

As original result, we also obtain the power spectrum of spatial Hawkes processes, including in our generalization a non-Poisson "ancestor process". Note that, although Hawkes processes are a particular case of cluster point processes, their spectrum cannot be obtained easily from the general formula of cluster point processes.

Random spike fields are defined in Chapter 1 and we recall the basic theory of Bartlett spectra in Chapter 3.

Filtered Random Spikes Fields

By filtered random spikes fields, we mean not only shot noises, but also cluster point processes (which can indeed be treated in a unified framework).

State of the Art. Shot noises have received much attention in the applied literature, whether in physics or in electrical engineering. They owe their name to the fact that they model, at the fine level, thermoionic noise in conductors [Campbell, 1909; Schottky, 1918], and they have been studied by Rice [1944] who contributed to their popularity (see also [Bondesson, 1988] and the references therein). The article of Lowen and Teich [1990] gives a number of application in physics (for instance, Cherenkov radiation) of the so-called *power–law* shot noise.

Shot noises appear in queuing, for instance under the form of a $M/GI/\infty$ pure delay system, which is a Poisson shot noise with random impulse function, and in traffic theory (see [Parulekar and Makowski, 1997] and the references therein). Shot noises have been used by Vere-Jones and Davies [1966] in the earthquake context and they are also of interest in insurance risk theory, where they represent delayed claims (see for instance [Klüppelberg and Mikosch, 1995; Samorodnitsky, 1995]).

The signals arising in neurophysiology are typically non-Poisson shot noises and the interference field in a mobile communication system is aptly modeled as a spatial shot noise (see for instance [Baccelli and Blaszczyszyn, 2001]). Shot noises also arise naturally in signal processing. For instance, the estimation of a signal with finite rate of innovation can be performed through regular sampling of a shot noise built from the signal [Vetterli et al., 2002; Maravic and Vetterli, 2004]. Another interesting example is provided by wavelet signal analysis when the analyzed signal is a point process [Abry and Flandrin, 1996], since the wavelet coefficients are in this case samples of shot noises. Wavelet statistical analysis has been proposed to detect and compute the Hurst parameter in classical signals [Abry and Veitch, 1998] and the method applies equally well to random Dirac combs with long-range dependence properties. The accuracy of the statistical analysis depends very much on the second order properties of the shot noises resulting from the wavelet decomposition.

Cluster point processes [Cox and Isham, 1980; Daley and Vere-Jones, 1988, 2002] appear in network traffic modeling [Latouche and Remiche, 2002; Hohn et al., 2003]. Existing works are limited to Poisson seeds, but the consideration of more general seed processes is indispensable for more realistic applications.

The power spectrum of shot noises and cluster point processes are known results. More precisely, Brémaud and Massoulié [2002] presented the spectrum of a shot noise on the real line with random impulse function and generic underlying point process, while the results on clustered point processes, jittered and thinned point processes appear in [Daley and Vere-Jones, 1988, 2002] (respectively Example 8.2 (d) and Exercise 8.2.6).

Original Contribution. One of our main results, given in Section 4.1, is the *fundamental isometry formula*. This general formula has a "swiss army knife" structure: by choosing the appropriate form of the formula one can derive the power spectrum of a variety of spatial processes, obtaining well known expression as well as new ones. For instance, such a formula provides the power spectrum of:

- The *spatial* shot noise with *random impulse function*, when the underlying stationary point process has a known Bartlett spectrum and the random impulses are independent and equally distributed and *independent* of the basic point process;
- The jittered and thinned point processes, that is general spatial stationary point processes with given Bartlett spectrum whose points are randomly and independently displaced or lost according to a vector of given distribution;
- The cluster point processes, with a general spatial stationary point process as basic point process, with given Bartlett spectrum, and independent and identically distributed (i.i.d.) clusters of finite point processes;

With respect to the existing results our contribution is to derive spectral expressions in the *spatial case* from a *single fundamental formula*.

In the fundamental isometry formula, the domain of the spectrum is described in detail in term of the domain of the spectrum of the underlying spike field. These details are missing in the literature.

The interest of a general abstract formula is that it provides a unifying and modular approach to the spectrum computation, allowing to progressively add various features to the basic model of a generic filtered spike field in a systematic way. Such a formula is then a key tool for obtaining various extensions of existing results and to provide spectra of more complex signals related to filtered random spikes.

A second result concerning filtered random spike fields is the power spectrum of generalized linear birth and death process (not necessarily Markovian). They are shot noises where the basic point process is a Hawkes process and the sequence of pulses is *not* independent of the basic point process.

Filtered fields are defined in Chapter 2 and the fundamental isometry formula is given in Chapter 4. Generalized linear birth and death processes and their power spectrum computation are presented in Chapter 6.

Modulated Random Spike Fields

A modulated random spike field is a Dirac comb with pulses of varying height. In random sampling, the height of a pulse is equal to the value of the signal sampled at this time. Random sampling has been extensively studied in view of spectral analysis, the object being to recover the power spectrum of the signal from the modulated sample comb, or even from the sample sequence (without timing information); a specific domain of application is laser velocimetry (see [Gaster and Roberts, 1975]), where the samples are collected only at the passage of a reflecting particle through the laser beam.

Two theoretical questions arise. The first one is related to spectral analysis, the second one to signal reconstruction.

- What is the relation between the spectrum of the modulated Dirac comb (or the sample sequence) to that of the signal?
- To what extent can we recover the signal from the modulated Dirac comb (or the sample sequence)?

State of the Art. Early investigation on random sampling [Shapiro and Silverman, 1960; Beutler, 1970] was mostly motivated by the search for alias-free sampling schemes, that is, sampling schemes leading to a one-to-one relation between the spectrum of the sample comb to that of the sampled signal.

The first detailed analysis of randomly sampled signals were based on the modeling of the sample comb using the Dirac (pseudo) process δ. Beutler and Leneman [1966], Leneman [1966] and Beutler [1968] obtained formulas for the moments of the sample comb that lead to the expression of the correlation of the sample comb as a function of the correlation of the sampled signal. Leneman and Lewis [1966] investigated the reconstruction error for several interpolators of the random samples. Such results depend on the sampling scheme through statistics related to the intervals between successive points of the sampler. Point processes for modeling randomly sampled processes were introduce in the path-laying paper of Brillinger [1972]

The spectrum of randomly sampled signals has been obtained in a general case but with strong conditions by Masry [1978b,a], using a point process approach. The spectrum of the sample sequence was expressed as a function of the spectrum of the sampled signal and of the second order quantities of the point process, and then, by reformulating the alias-free concept, alias-free sampling schemes were proved to lead to a consistent spectral estimator.

The power spectrum of a modulated spike field with a general underlying point process appears in [Daley and Vere-Jones, 1988, 2002].

Original Contribution. Our work is close to Masry [1978b,a], and our method of proof is the same as in [Daley and Vere-Jones, 1988, 2002]. Our contribution is to give details for the informal proof of Daley and Vere-Jones [1988, 2002], details which are useful to determine the class of test functions for which the defining formula of the spectrum is true. Also, we give the power spectra of modulated spike fields when the sampler is possibly dependent on the signal. In the independent case, we also give the expression of the error when the signal is approximated by a filtered version of the samples, that is, the reconstruction error, unifying under the same framework various results scattered in the literature.

Modulated random spike fields are defined in Chapter 2 and their power spectrum formula is presented in Chapter 5. Related results are presented in Chapter 9.

Complex Signals

As already mentioned, the power spectrum of complex signals can be obtained as the results of various operations on the basic models of spike fields, filtered random spike fields and modulated random spike fields.

Here, we turn our attention to signals arising in network traffic and to the large family of signals that characterize Ultrawide Bandwidth (UWB) communications, namely pulse modulated signals, direct sequences, time-hopping signals and multipath faded pulse trains.

State of the Art. Poisson cluster models have recently been proposed as a natural model for packets arrivals in network traffic models [Hohn et al., 2003]. The expression of their power spectrum then provides a tool for traffic analysis, particularly useful in the study of long-range properties [Hohn et al., 2003; Hohn and Veitch, 2003]. As already mentioned, the spectrum of a cluster point process is a known result [Daley and Vere-Jones, 1988, 2002].

Concerning pulse modulated signals, their exact spectral evaluation has already received the attention in the communications community. Among several contributions, we mention the computation of the power spectrum of a general time hopping, pulse position modulated signal in the presence of clock jitter [Win and Scholtz, 1998; Win, 2002; Romme and Piazzo, 2002], and that of the spectral density of the family of pulse interval modulated signals [Cariolaro et al., 2001].

In these contributions, the signal models are based on Dirac pseudo-functions and lack generality, especially with respect to the type of temporal modulation. The spectrum computation is performed using the classical w.s.s. approach, *i.e.*, as the Fourier transform of the correlation function. Although this is a common approach, the resulting computations are complex and lack mathematical rigour: it is then particularly difficult to establish the exact domains of validity of the results. Moreover, the introduction of additional random quantities requires a new computation from scratch.

A shot noise perspective of the output of multipath fading channels has been proposed by Charalambous et al. [2001]. It consists in a macroscopic model that does not allow a fine characterization of the multipaths since the random instants of the transmitted pulses and of the reflected ones are undistincly modeled with the same point process. Moreover, the model is limited to specific basic point processes and specific filtering functions, and the computation of the spectral properties is restrained to the Poisson case.

Original Contribution. The spectrum of cluster point process is derived within the unifying framework of the fundamental isometry formula. Such an approach allows to consider more complex cluster models (*e.g.*, double cluster models in the presence of random perturbations such as losses, jitter and distortions), allowing for various extension of existing results.

Concerning UWB signal, we exploit the fact that they can be viewed as shot noises, and we derive a unifying model that includes UWB pulse modulations and general UWB signaling, such as direct sequences and time-hopping. Their power spectrum can be then derived using the spectral formula of a shot noise with random excitation.

We also consider the exact spectral evaluation of a pulse transmission over a multipath fading channel. The model we propose is very general, simple and tractable, and, from the point process perspective, it corresponds to a shot noise with random excitation, modulated by a w.s.s. process. It allows to finely account for various phenomena that affects the transmission, such as the multipaths, jitter, attenuation, losses and distortion of the pulses, and, as a special case, it provides the classical double Poisson model of Saleh and Valenzuela [1987]. To the best of our knowledge, the general model we propose and its exact spectral evaluation are novel results.

In summary, by mean of the fundamental isometry formula, we provide a general, simpler, systematic and rigorous approach to the evaluation of the spectra of complex signals. Moreover, our approach is modular. In fact, we start from a general formula giving the spectrum of a generalized shot noise (non Poisson) with a random impulse function, and build progressively on it to add specific features of the model in a systematic manner. In addition, the exact power spectrum formula we obtain is easy to understand since the contribution corresponding to various features of the model *appears clearly and separately* in the power spectrum expression.

Outline

The first two chapters mainly consist of definitions, and they present the basic items and the basic operations that are used to construct complex signals. In Chapter 1 we introduce random spike fields in mathematical terms, using the point process formalism, while in Chapter 2 we describe the basic operations on random spike fields, namely, filtering, jittering, thinning, clustering and modulation.

In Chapter 3 we recall the classical spectral theory for wide-sense stationary processes and then we present the Bartlett spectral theory for point processes, discussing, case by case, the domain of definition.

The next two chapters present the spectrum formulas that constitute the "toolbox" for the computation of the spectra of complex signals: the fundamental isometry formula, which is the "seed" formula (Chapter 4), and its extensions to the case where the point processes are modulated (Chapter 5).

Chapter 6 concerns the spectra of Hawkes processes and the generalized birth and death processes. The latter are shot noises were the basic point process is a Hawkes process and the shots are not independent of the basic process, and for which the spectrum "toolbox" does not apply.

Applications are presented in Chapters 7, 8 and 9. More precisely, we provide very general models and the corresponding general spectrum formulas for UWB signals, pulse trains over multipath fading channels and randomly sampled signals.

Chapter 1

Random Spikes

Summary: This chapter mainly consists of definitions. It introduces the random spike fields in mathematical terms using the point process formalism.

Point processes aptly model sequences of time events or spatial configurations of points. As we shall see, they can be considered as a stream of spikes at random positions or as a sequence of random times.

This chapter mainly consists of definitions. The monographs of Neveu [1977] and Daley and Vere-Jones [1988, 2002] are the basic references on point processes, especially for the spectral theory.

1.1 Point Processes

Definition 1.1: **Point Process**.

A simple, locally finite, spatial point process is a random set N of $\mathbb{R}^m$ such that for each bounded Borel set $A \subset \mathbb{R}^m$, the random variable

$$N(A) := |N \cap A|$$

(where $|C|$ =cardinal of C) is almost surely finite.

On the real line, this random set can be represented uniquely by a random sequence of $\bar{\mathbb{R}}$, $\{T_n\}_{n\in\mathbb{Z}}$, such that

$$T_0 \le 0 < T_1 \text{ (convention)}, \quad |T_n| < \infty \Rightarrow T_n < T_{n+1}, \\ \lim_{n\uparrow\infty} T_n = +\infty, \quad \lim_{n\downarrow-\infty} T_n = -\infty.$$

Hence, in this case

$$N = \{T_n;\, |T_n| < \infty,\, n \in \mathbb{Z}\}.$$

If $\mathrm{P}(N(\mathbb{R}^m) < \infty) = 1$, the point process is called *finite*. ◊

A point process can also be described by the random *counting measure*

$$N(A) = \sum_{s\in N} 1_A(s), \quad A \subset \mathbb{R}^m,$$

indicating the (random) number of points falling in the set A (N is used to denote both the random set and the random counting measure).

Symbolically, we can think of a point process as a sum of Dirac pseudo functions $\delta(t)$ or spikes, centered at the random instants $s \in N$, *i.e.*, a pseudo function

$$\Delta_N(t) := \sum_{s \in N} \delta(t-s), \quad t \in \mathbb{R}^m.$$

Figure 1.1 depicts a realization of the counting measure on the real line.

Using the counting measure formalism, the following three expressions represent the same object

$$\int_{\mathbb{R}^m} \varphi(t) N(dt) = \int_{\mathbb{R}^m} \varphi(t) \Delta_N(t)\, dt = \sum_{s \in N} \varphi(s).$$

The first expression is an integral with respect to the measure N; the second one is symbolic, and uses the symbolic rule $\varphi(0) = \int_{\mathbb{R}} \varphi(t)\delta(t)dt$. These quantities are well-defined only under certain circumstances, for instance when $\varphi \geq 0$ (in which case it may be infinite). More precisely, exact conditions for the definition of the above expression are provided by the Campbell theorem (see for instance [Daley and Vere-Jones, 1988, 2002] or Definition 1.4).

In the following, for simplicity, we shall sometimes adopt the notation

$$\eta(\varphi) := \int_{\mathbb{R}^m} \varphi(t) \eta(dt),$$

where η is a general measure (deterministic or random). In particular

$$N(\varphi) := \int_{\mathbb{R}^m} \varphi(t) N(dt). \tag{1.1}$$

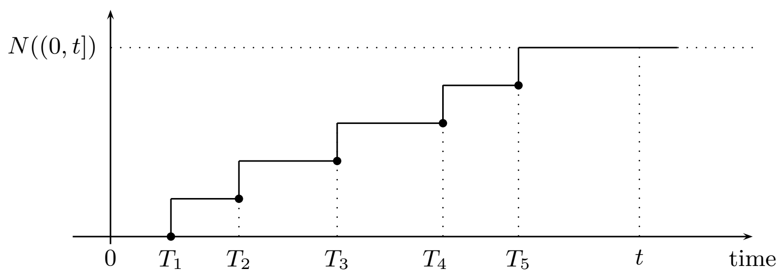


Figure 1.1: The counting measure N on the real line.

Definition 1.2: **Stationary point process**. A point process N on $\mathbb{R}^m$ is said to be stationary if for every $k = 1, 2, \dots$ and all bounded Borel subsets $A_1, \dots, A_k$ of $\mathbb{R}^m$, the joint distribution of

$$\{N(A_1 + s), \dots, N(A_k + s)\}$$

does not depend on $s \in \mathbb{R}^m$. ♢

We now give some examples of point processes that are of interest in this work.

EXAMPLE 1.1: Renewal point processes. Recall the convention $T_0 \leq 0 < T_1$ (Definition 1.1). Renewal point processes are a particular case of point processes on the real line, where the random variables

$$S_n := T_{n+1} - T_n, \quad n \in \mathbb{Z}/\{0\}$$

called the *inter-arrival times*, are i.i.d. and independent of (T_0, T_1). In this context, the sequence $\{T_n\}_{n\in\mathbb{Z}}$ is called the *renewal sequence* and each of its terms *renewal epochs*.

The process is called *delayed* if $\mathrm{P}(T_0 < 0) > 0$; *pure* if $T_0 = 0$, and T_1 has the common distribution of $\{S_n\}_{n\in\mathbb{Z}/\{0\}}$. In the pure case, the origin is a renewal epoch.

A pure renewal process, is inevitably non stationary, while a delayed renewal process is stationary if $-T_0$ and T_1 are identically distributed with common distribution

$$F(t) = \lambda \int_0^t (1 - F_S(u))\, du\,,$$

where F_S is the common distribution of $\{S_n\}_{n\in\mathbb{Z}/\{0\}}$ (see for instance [Daley and Vere-Jones, 1988, 2002; Baccelli and Brémaud, 2003]).

Figure 1.2 depicts a pure and a delayed renewal processes.

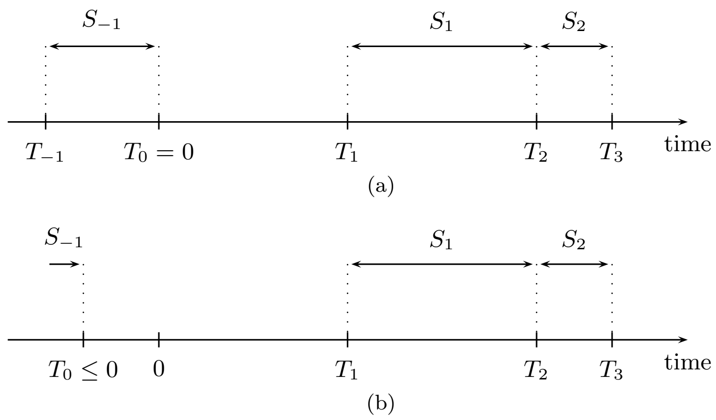


Figure 1.2: Renewal processes: Pure (a) and Delayed (b).

Renewal processes are widely used in the modeling of network traffic, where they represent the arrival times of packets (see for instance [Hohn et al., 2003]), and in pulse interval modulations, where the inter-arrival times represent the support for the information (see for instance [Ghassemlooy et al., 1998]).

EXAMPLE 1.2: Uniformly spaced points on the plane (regular grid). A regular (T_1, T_2)-grid on $\mathbb{R}^2$ can be modelled as a point process

$$N = \left\{ (n_1 T_1 + U_1, n_2 T_1 + U_2), (n_1, n_2) \in \mathbb{Z}^2 \right\}, \tag{1.2}$$

where $T_1 > 0$, $T_2 > 0$, and U_1, U_2 are independent random variables characterizing the random origin of the grid. Regular grids are a particular case of a double renewal process. When $U_1 \equiv U_2 \equiv 0$ we have a pure renewal process, and the two sequences of inter-arrival times have constant values equal to T_1 and T_2, respectively. Otherwise, we have a delayed double renewal process, and choosing U_1, U_2 as uniformly distributed, respectively on $[0, T_1]$ and $[0, T_2]$, guarantees the stationarity of the process.

Definition 1.3: **First and second order point processes**. Let N be a *simple and locally bounded* point process on $\mathbb{R}^m$. It is called a *first order* point process if for all bounded Borel sets $C \subset \mathbb{R}^m$,

$$\mathrm{E}\left[N(C)\right] < \infty\,. \tag{1.3}$$

It is called a *second order* point process if for all bounded Borel sets $C \subset \mathbb{R}^m$

$$\mathrm{E}\left[N\left(C\right)^2\right] < \infty\,. \tag{1.4}$$

◊

Definition 1.4: **Intensity**. If N is a first order point process, then

$$\eta(C) = \mathrm{E}\left[N(C)\right] \tag{1.5}$$

defines a Radon (that is, locally finite) measure η on $\mathbb{R}^m$, called the *mean measure*, or *intensity measure* of N.
By Campbell's theorem, for all measurable functions $\varphi : \mathbb{R}^m \to \mathbb{R}$, that are nonnegative or such that $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\eta)$, *i.e.*, $\int_{\mathbb{R}^m} \left|\varphi\left(s\right)\right| \eta\left(ds\right) < \infty$, the sum

$$\sum_{t\in N} \varphi(t) = \int_{\mathbb{R}^m} \varphi\left(s\right) N\left(ds\right)$$

is well-defined and

$$\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi\left(s\right) N\left(ds\right)\right] = \int_{\mathbb{R}^m} \varphi\left(s\right) \eta\left(ds\right)\,. \tag{1.6}$$

When the intensity measure admits a density, *i.e.*,

$$\eta(ds) = \lambda\left(s\right) ds\,,$$

we call $\lambda\left(s\right)$ the *intensity* of the point process N. Notice that when the density is constant, $\mathrm{L}^1_{\mathbb{C}}(\eta) = \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$.

◊

EXAMPLE 1.3: Poisson point process. A Poisson point process is a particular case of renewal point process where the inter-arrival times are exponentially distributed. More precisely

$$f_{S_n}\left(s\right) = \lambda e^{-\lambda s}, \quad n \in \mathbb{Z}/\left\{0\right\}\,,$$

where λ is the intensity of the process.

In the delayed stationary case we have

$$f_{T_1}\left(s\right) = f_{-T_0}\left(s\right) = \lambda e^{-\lambda s}\,.$$

Moreover, if the intensity is constant, the Poisson process is said to be *homogeneous*.

EXAMPLE 1.4: Cox process. A Cox point process on $\mathbb{R}^m$, with stochastic intensity $\left\{\lambda(t)\right\}_{t\in\mathbb{R}^m}$ is a point process N such that

- $\left\{\lambda(t)\right\}_{t\in\mathbb{R}^m}$ is a nonnegative a.s. locally integrable stochastic process;
- *conditionally on* $\lambda(t)$, N is a Poisson process with intensity $\lambda(t)$.

A Cox process is stationary if and only if the stochastic intensity is stationary.

1.2 Marked Point Processes

Modeling often requires the introduction of a sequence of *marks* associated to the random points of N.

Definition 1.5: **Marked Point Process**. Let $(K, \mathcal{K})$ be some measurable space and let

$$\bar{N} = \{(t, z) \in \mathbb{R}^m \times K\}$$

be a random set in $\mathbb{R}^m \times K$ such that the random set in $\mathbb{R}^m$

$$N = \left\{t \in \mathbb{R}^m ; \exists\ z \in K \text{ s.t. } (t, z) \in \bar{N}\right\}$$

is a locally finite and simple point process. N is called the *basic point process* of $\bar{N}$, and the latter is called a *marked point process* on $\mathbb{R}^m \times K$ *with locally finite and simple basic point process.*

Define a measure on $\mathbb{R}^m \times K$, also denoted by $\bar{N}$, as follows

$$\bar{N}(C \times L) = \sum_{(t,z)\in \bar{N}} 1_C(t) 1_L(z) . \tag{1.7}$$

Then, for all measurable function $\varphi : \mathbb{R}^m \times K \to \mathbb{R}$,

$$\int\limits_{\mathbb{R}^m \times K} \varphi(t, z) \bar{N}(dt \times dz) = \sum_{(t,z)\in \bar{N}} \varphi(t, z) ,$$

provided that the sum in the right-hand side is well-defined. ◊

A particular case is when the marks are i.i.d., and independent of the basic point process.

Definition 1.6: **Marked Point Process with i.i.i.d. marks**. Let N be a simple, locally finite, point process and $\{Z(t)\}_{t\in\mathbb{R}^m}$, be an i.i.d. family of K-valued random variables, that are independent of N. Define the random set

$$\bar{N} = \{(t, Z(t)) ;\ t \in N\} ,$$

or

$$\bar{N}(C \times L) = \sum_{t\in N} 1_C(t)\, 1_L(Z(t)) , \quad C \subset \mathbb{R}^m , \quad L \subset K ,$$

with the notation

$$\int_{\mathbb{R}^m} \varphi(t, Z(t))\, N(dt) = \sum_{t\in N} \varphi(t, Z(t)) ,$$

provided that the sum in the right-hand side is well defined. Then $\bar{N}$ is called a (simple, locally finite) marked point process, with i.i.d. marks in K independent of N. For short, we shall say that $\bar{N}$ is a marked point process on $\mathbb{R}^m$ with i.i.i.d. marks in K (note the triple “i”). We call $\{Z(t)\}_{t\in\mathbb{R}^m}$ its mark process. ◊

For a point process on the real line ($\mathbb{R}^m = \mathbb{R}$), we have

$$\bar{N}(C \times L) = \sum_{n\in\mathbb{Z}} 1_C(T_n) 1_L(Z_n) ,$$

$$\int\limits_{\mathbb{R}} \varphi(t, Z(t)) N(dt) = \sum_{n\in\mathbb{Z}} \varphi(T_n, Z_n) ,$$

(where the sum extend to all n such that $|T_n| < \infty$).

In general, the mark process and the basic counting process are not defined independently. One may start with a point process $\bar{N}$ on $\mathbb{R}^m \times K$ with locally finite and simple basic point process, and define the mark process $\{Z(t)\}_{t\in\mathbb{R}^m}$, K-valued as follows:

$$\forall\ t \in \mathbb{R}^m, \quad Z(t) = \begin{cases} z & \text{if } (t,z) \in \bar{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The process $\{Z(t)\}_{t\in\mathbb{R}^m}$ is the mark process associated with N. Of course

$$\sum_{(t,z)\in\bar{N}} \varphi(t,z) = \sum_{t\in N} \varphi(t, Z(t))\,. \tag{1.8}$$

Chapter 2

Operations on Spikes

Summary: This chapter presents the processes that are obtained by the basic operations of filtering, jittering, thinning, clustering and modulation on point processes. It mainly consists of definitions.

Jittering, thinning, filtering, clustering and modulating are basic operations on point processes. As we shall see in Chapters 7, 8, and 9, combining and nesting such operations leads to several complex signals of interest in communications.

2.1 Jittering

Jittering is a common term for describing random displacement of time events. In particular, we say that a point process is "jittered" when its points have been randomly displaced.

When the random displacements are i.i.d., a jittered point process is aptly modeled by mean of a marked point process by considering the following setup:
Let N be a (simple, locally finite) point process on $\mathbb{R}^m$ and $\bar{N}$ be a marked point process with basic point process N and i.i.i.d. marks $\{Z(t)\}_{t\in\mathbb{R}^m}$ taking values over $\mathbb{R}^m$. Call $\widetilde{N}$ the jittered point process obtained by randomly displace the points of N. Then,

$$\widetilde{N}(C) = \sum_{t\in N} 1_C(t + Z(t)) \sum_{(t,z)\in\bar{N}} 1_C(t+z)$$

Figure 2.1 depicts a jittered point process on the real line.

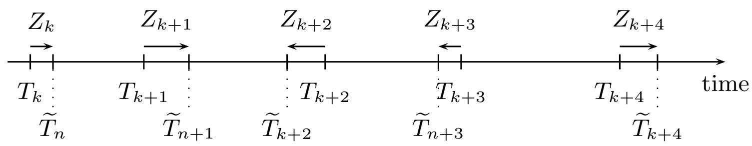


Figure 2.1: A jittered point process on the real line: T_k denotes the points of the original (basic) process N, while $\widetilde{T}_n$ denotes the points of the jittered process $\widetilde{N}$.

2.2 Thinning

Consider a point process N (simple, locally finite) affected by random i.i.d. losses (thinning) of its points. Such a situation can be easily modeled using a marked point process $\bar{N}$, with basic point process N and $\{0,1\}$ valued i.i.i.d. marks.

The thinned point process $\widetilde{N}$ then reads

$$\widetilde{N}(C) = \bar{N}(C \times \{1\}) = \sum_{t \in N} 1_C(t)\, 1_{\{1\}}(Z(t)) = \sum_{t \in N} 1_C(t)\, Z(t) \ .$$

Figure 2.2 depicts a thinned point process on the real line.

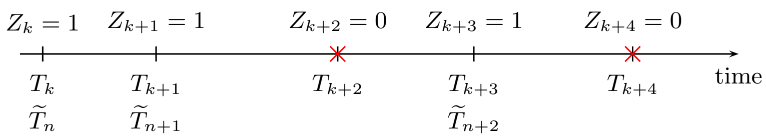


Figure 2.2: A thinned point process on the real line: T_k denotes the points of the original (basic) process N, while $\widetilde{T}_n$ denotes the points of the thinned process $\widetilde{N}$.

2.3 Filtering

When the point process is fed into a filter with impulse response h, we obtain the so called *shot noise*.

Symbolically, we can think of the input as a stream of delta functions, or "Dirac comb", $\Delta_N(t)$. The output is then

$$X(t) = \int_{\mathbb{R}} h(t-s)\, \Delta_N(s) ds = \sum_{n \in \mathbb{N}} h(t - T_n) \ .$$

Such a notion can be extended to the spatial case:

Definition 2.1: **Shot noise**. Let $h : \mathbb{R}^m \to \mathbb{R}$ be a measurable function and N a simple, locally finite point process. The process $\{X(t)\}_{t \in \mathbb{R}^m}$ defined (when possible) by

$$X(t) = \int_{\mathbb{R}^m} h(t-s) N(ds) = \sum_{s \in N} h(t-s) \tag{2.1}$$

is called a *spatial shot noise*. ◊

By Campbell's theorem, when N is a first order point process with intensity measure η, and $h \in \mathrm{L}^1_{\mathbb{C}}(\eta)$, the shot noise process (2.1) is well defined (recall that η is the intensity measure of N; see Definition 1.4).

EXAMPLE 2.1: Exponential shot noise. Let N be any point process and consider the exponential impulse response

$$h(t) = \beta e^{-\alpha t} 1_{\{t \geq 0\}} \ ,$$

where β, α are positive real numbers. The corresponding shot noise is

$$X(t) = \beta \int_{-\infty}^{t} e^{-\alpha(t-s)} N(ds) \, .$$

Of particular interest is the situation where the impulse response of the filter depends on a random parameter. We then have a *shot noise with random excitation*.

Definition 2.2: **Shot noise with random excitation**. Let $(K, \mathcal{K})$ be some measurable space, $h : \mathbb{R}^m \times K \to \mathbb{R}$ be a measurable function, and $\bar{N}$ a simple, locally finite marked point process with K-valued i.i.i.d. marks (see Definition 1.6). The process $\{X(t)\}_{t\in\mathbb{R}^m}$ defined (when possible) by

$$X(t) = \int_{\mathbb{R}^m \times K} h(t-s, z)\bar{N}(ds \times dz) = \sum_{(s,z)\in\bar{N}} h(t-s, z) \tag{2.2}$$

is called a *spatial shot noise with random excitation.* ♢

By Campbell's theorem, the shot noise with random excitation (2.2) is well defined if N is a first order point process and $h \in \mathrm{L}^1_{\mathbb{C}}(\eta \times Q)$, where Q is the common distribution of the marks. We recall that $\mathrm{L}^p_{\mathbb{C}}(\eta \times Q)$ is the set of functions $\varphi : \mathbb{R}^m \times K \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^m} \mathrm{E}\left[|\varphi(t, Z)|^p\right] d\eta < \infty \,,$$

where here η is the intensity measure (see Definition 1.4).

EXAMPLE 2.2: Shot noise with random losses (thinning). A filtered point process affected by i.i.d. random losses is aptly modelled with a shot noise with random excitation by taking as random filtering function

$$h\left(t, Z\right) := Zh\left(t\right) \,,$$

where now $E = \{0, 1\}$, *i.e.*, the marks form a sequence of $\{0, 1\}$ i.i.d. random variables.

An extension of the previous examples is the following:

EXAMPLE 2.3: Shot noise with binary amplitudes - amplitude modulation. We now consider N to be a regular T-grid on the line and $E = \{-1, 1\}$. Then, a shot noise with random filtering function given by

$$h\left(t, Z\right) := Zh\left(t\right) \,,$$

represent the basic model of a pulse amplitude modulation, with i.i.d. amplitudes (see Figure 2.3).

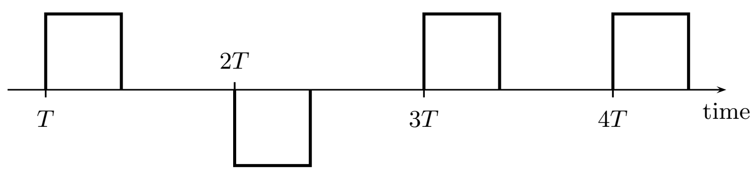


Figure 2.3: The basic pulse amplitude modulation.

EXAMPLE 2.4: Shot noise with random displacements (jitter). Similarly as in the case of random losses, random displacements is introduced by mean of the marks. The random filtering function is now

$$h(t, Z) := h(t - Z).$$

Figure 2.4 depicts a shot noise affected by random displacements.

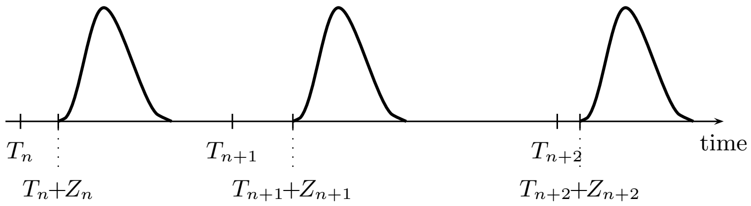


Figure 2.4: Shot noise with random displacements.

EXAMPLE 2.5: Spatial $M/GI/\infty$ process. Let $\bar{N}$ be a marked point process where the basic point process is Poisson on $\mathbb{R}^2$, with intensity λ, and the i.i.d. marks $\{Z(t)\}_{t\in\mathbb{R}^2}$ are random subset of $\mathbb{R}^2$ (to avoid formal definitions, think of simple shapes, for example disks of random radius centered at 0). Then, the shot noise

$$X(t) = \sum_{(s,z)\in\bar{N}} h(t-s, z) = \sum_{s\in N} 1_{\{t\in(s+Z(s))\}},$$

with

$$h(t, z) = \begin{cases} 1 & \text{if } t \in z, \\ 0 & \text{otherwise}, \end{cases}$$

is a spatial $M/GI/\infty$ process. It represents the number of "shapes" $s + Z(s)$ centered at $s \in N$ that *cover* t. Figure 2.5 depicts the spatial $M/GI/\infty$ process.

EXAMPLE 2.6: Interferences in wireless communication network. In mobile communication systems, shot noises with random excitation allows to model interferences between emitting mobile units. Now,

- the basic point process N on $\mathbb{R}^2$ represents the locations of the mobile units;
- $h(t, z) = zl(t)$, where $l : \mathbb{R}^2 \to \mathbb{R}_+$ gives the *attenuation* function or *path-loss* of the emitting signal, and $z \in \mathbb{R}_+$ represents the power of the signal emitted.

Then, the shot noise

$$X(y) = \sum_{s\in N} Z(s)\, l(y-s)$$

represents the configuration of the emission power of the mobile units. In particular, a point $y \in \mathbb{R}^2$ receives the the signal of the station located at $u \in \mathbb{R}^2$ at a level $Z(u)\, l(y-u)$, and,

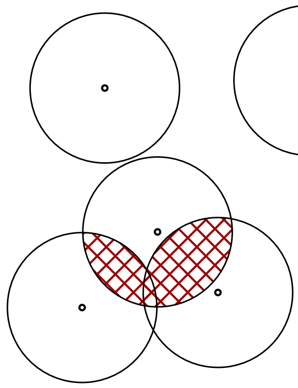


Figure 2.5: Spatial $M/GI/\infty$ process.

with respect to such signal, the other stations perturb the reception with a "noise" given by their transmissions

$$W_u(y) = \sum_{s \in N, s \neq u} Z(s)\, l(y-s) \ .$$

We can then fix a threshold θ and ask that, in order to reliably receive the signal, the level or reception is θ times bigger that the surrounding noise W. This type of model was proposed by Baccelli and Blaszczyszyn [2001] to compute the configuration of the region where each station can reliably communicate.

2.4 Clustering

We give the definition of a general class of cluster point processes. We first consider a cluster point process on the real line and then extend it to the case of $\mathbb{R}^n$.

Let N be a simple, locally finite, stationary point process on $\mathbb{R}$, in this context called *the seed*, with intensity λ, $0 < \lambda < \infty$, and sequence of points $\{T_n\}_{n\in\mathbb{Z}}$ (recall the convention $T_0 \leq 0 < T_1$). Let $\{Z_n\}_{n\in\mathbb{Z}}$ be an i.i.d. sequence of simple point processes on $\mathbb{R}_+$ and independent of the seed N.

The cluster point process N_c (based on N, with cluster sequence $\{Z_n\}_{n\in\mathbb{Z}}$) is defined by

$$N_c(C) = \sum_{n\in\mathbb{Z}} Z_n(C - T_n) \ , \quad \forall\, C \subset \mathbb{R} \, .$$

Call Z a point process distributed as the common distribution of $\{Z_n\}_{n\in\mathbb{Z}}$. We suppose that $\mathrm{E}[Z(\mathbb{R}_+)] < \infty$, so that the simple point process N_c has intensity $\lambda \mathrm{E}[Z(\mathbb{R}_+)] < \infty$.

EXAMPLE 2.7: Renewal clusters. In this example, the typical cluster is a random piece of a renewal process. Such models were introduced by Cox and Isham [1980]. Denoting by ε_a the Dirac measure at $a \in \mathbb{R}$

$$Z_n = \sum_{l=1}^{L_n} \varepsilon_{t_l^{(n)}} \ , \quad n \in \mathbb{Z} \, ,$$

where $t_0^{(n)} \equiv 0$, $t_l^{(n)} = S_1^{(n)} + \ldots + S_l^{(n)}$ $(l \geq 1)$, L_n is an integer-valued random variables, and

- L_n is independent of $\left\{S_k^{(n)}\right\}_{k\geq 1}$;
- $S^{(n)} \stackrel{\text{def}}{=} \left\{S_k^{(n)}\right\}_{k\geq 1}$ is i.i.d., with c.d.f. F such that $F(\{0\}) = 0$;
- the sequence $\left\{\left(L_n, S^{(n)}\right)\right\}_{n\in\mathbb{Z}}$ is i.i.d..

Then,

$$N_\mathrm{c} = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{L_n} \varepsilon_{T_n + t_l^{(n)}} \,. \tag{2.3}$$

Call L a random variable distributed as the common distribution of $\{L_n\}_{n\in\mathbb{Z}}$. We suppose that $\mathrm{E}[L] < \infty$. Therefore the cluster point process has intensity $\lambda\mathrm{E}[L] < \infty$ and it is simple.

Renewal cluster point processes can be extended to $\mathbb{R}^n$ by considering a sequence of point in $\mathbb{R}^n$ connected by random vectors.

An interesting application of renewal cluster point process is the modeling of network traffic (see for instance [Hohn et al., 2003]).

The model is easily extended to spatial and more general point processes.

Definition 2.3: **Cluster point process**. Consider a marked point process with i.i.i.d. marks taking values on some measurable space $(K, \mathcal{K})$ (see Definition 1.6). Here we suppose that $K = M_p(E)$ is the space of the random (locally finite) measures μ on $(\mathbb{R}^m, \mathcal{B}^m)$ such that $\mu(C) \in \mathbb{N}$ for all relatively compact $C \in \mathcal{E}$, and that $\mathcal{M}_p(E)$ is the σ-field on $M_p(E)$ generated by the mappings $\mu \to \mu(C)$, $C \in \mathcal{E}$. Therefore, we suppose that for each $x \in E$, $Z(x)$ is a point process on $(\mathbb{R}^m, \mathcal{B}^m)$, and we assume that it is a simple one (without multiple points), with a finite average number of points. We denote in this particular situation

$$Z(x)(C) = Z(x, C) \ , \quad C \in \mathcal{E}$$

(we remark that, with respect to the notation used for the definition of a cluster point process on the real line, here we have $Z(x) = A_x$, $x \in E$). The cluster point process N_c attached with N and $\{A_x\}_{x\in E}$ is defined by

$$N_\mathrm{c}(C) = \sum_{x\in N} Z(x, C - x) \ , \tag{2.4}$$

for all $C \in \mathcal{E}$.

We shall also consider the point process

$$N'_\mathrm{c}(C) = N(C) + \sum_{x\in N} Z(x, C - x) \ , \tag{2.5}$$

which corresponds to the case where the seeds are systematically counted as points of the cluster point process. ◊

We remark that the renewal point process of Example 2.7 corresponds to a cluster point process of the form (2.5). Indeed

$$\sum_{n\in\mathbb{Z}} \sum_{l=0}^{L_n} \varepsilon_{T_n + t_l^{(n)}} = N + \sum_{n\in\mathbb{Z}} \sum_{l=1}^{L_n} \varepsilon_{T_n + t_l^{(n)}} \ ,$$

since $t_l^{(n)} \equiv 0$.

2.5 Modulation

In a modulated random spike field, the amplitudes of the spikes vary according to a certain stochastic process. We shall consider two situations: a point process modulated by a continuous time process or by a time series.

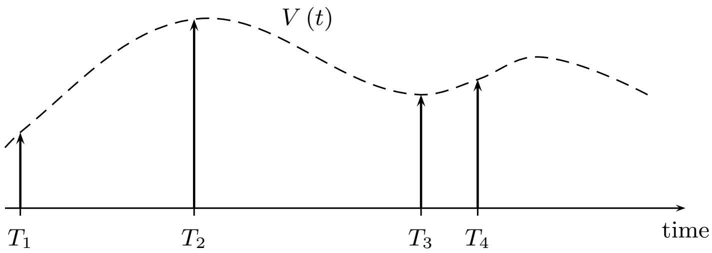


Figure 2.6: Point process modulated by a continuous time process.

Modulation by a Continuous Time Process

Let $\{V(t)\}_{t\in\mathbb{R}^m}$ be a stochastic process and N a simple, locally finite, point process on $\mathbb{R}^m$. Using the Dirac symbolism, a modulated point process can be seen as the generalized process

$$Y(t) = \sum_{s\in N} V(s)\,\delta(t-s)\ , \quad t\in\mathbb{R}^m\ .$$

EXAMPLE 2.8: Random sampling. Random sampling of a continuous time random signal $\{V(t)\}_{t\in\mathbb{R}}$, yields a sequence of samples

$$\{V(T_n)\}_{n\in\mathbb{Z}}\ ,$$

where $\{T_n\}_{n\in\mathbb{Z}}$, is the sequence of points (times of events) of a point process. The randomly sampled signal is given by the generalized process

$$Y(t) = \sum_{n\in\mathbb{Z}} V(T_n)\,\delta(t-T_n)\ , \quad t\in\mathbb{R}^m\ .$$

We can extend the definition of a shot noise with random excitation to the situation where the basic point process is a modulated one.

Definition 2.4: Let $\{V(t)\}_{t\in\mathbb{R}^m}$ be a stochastic process, $(K,\mathcal{K})$ be some measurable space, $h:\mathbb{R}^m\times K\to\mathbb{R}$ be a measurable function, and $\bar{N}$ be a simple, locally finite marked point process with K-valued i.i.i.d. marks. The process $\{X(t)\}_{t\in\mathbb{R}^m}$ defined (when possible) by

$$X(t) = \int_{\mathbb{R}^m\times K} h(t-s,z)V(s)\,\bar{N}(ds\times dz) = \sum_{(s,z)\in\bar{N}} h(t-s,z)\,, \quad t\in\mathbb{R}^m\,, \qquad (2.6)$$

is a spatial shot noise with random excitation and modulated basic point process. ◊

The modulated shot noise with random excitation is well defined by Campbell's theorem under the conditions of N being a first order point process and, for almost all t, $\mathrm{E}\left[\left|h\left(t, Z\right) V\left(t\right)\right|\right] \in \mathrm{L}^2_{\mathbb{C}}(\eta)$ (recall that η is the intensity measure of N; see Definition 1.4).

One can straightforwardly check that when the modulating process is w.s.s. and independent of N, the shot noise with random excitation and modulated basic point process is well defined under the same condition stated for the non modulated case.

Modulation by a Time Series

We shall also consider the case where a point process on the real line is modulated by a time series, as depicted in Figure 2.7. Let $\{A_n\}_{n\in\mathbb{Z}}$ be a time series and N a simple, locally finite, point process on the real line ($\mathbb{R}$). Using the Dirac formalism, we define a modulated point process as the generalized process

$$Y\left(t\right) = \sum_{n\in\mathbb{Z}} A_n \delta(t - T_n), \quad t \in \mathbb{R}\,.$$

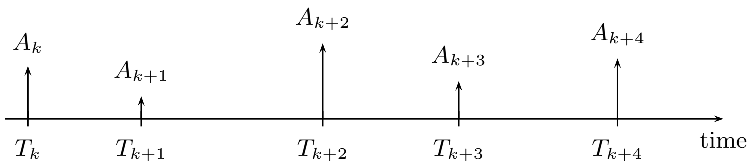


Figure 2.7: Point process modulated by a time series.

Definition 2.5: Let $\{A_n\}_{n\in\mathbb{Z}}$ be a stochastic process, $(K, \mathcal{K})$ be some measurable space, $h : \mathbb{R} \times K \to \mathbb{R}$ be a measurable function, and $\bar{N}$ be a simple, locally finite marked point process with K-valued i.i.i.d. marks. The process $\{X(t)\}_{t\in\mathbb{R}}$ defined (when possible) by

$$X(t) = \sum_{n\in\mathbb{Z}} A_n h\left(t - T_n, Z_n\right)\,, \quad t \in \mathbb{R}\,. \tag{2.7}$$

is a shot noise with random excitation and basic point process modulated by the time series $\{A_n\}_{n\in\mathbb{Z}}$. ◊

One can straightforwardly check that when the time series is w.s.s. and independent of N, the shot noise with random excitation and modulated basic point process is well defined under the same condition stated for the non modulated case.

Chapter 3

Bartlett Spectrum

Summary: We recall the classical spectral theory for wide-sense stationary processes and then introduce the Bartlett power spectrum for point processes.

Our contribution: We provide, with more precision than the general theory, the domain of definition of the Bartlett power spectrum.

3.1 Classical Wide-Sense Stationary Framework

3.1.1 The Bochner Power Spectrum

We recall that the spectral measure of a wide-sense stationary (w.s.s.) process is defined by the Bochner theorem ([Bochner, 1955], and see for instance [Dacunha-Castelle and Duflo, 1986]):

Theorem 3.1.1 (Bochner) *Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a w.s.s. process with continuous autocorrelation function $C_X : \mathbb{R}^m \to \mathbb{C}$. There exists a unique bounded measure μ_X on $\mathbb{R}$, called the power spectral measure, such that*

$$C_X(\tau) = \int_{\mathbb{R}^m} e^{i2\pi<\tau,\nu>} \mu_X(d\nu) , \quad \textit{for all} \quad \tau \in \mathbb{R} . \tag{3.1}$$

Moreover, if C_X is integrable, then the power spectral measure admits a density S_X, i.e., $\mu_X(d\nu) = S_X(\nu)\, d\nu$, and S_X is the Fourier transform of C_X, i.e.,

$$S_X(\nu) = \int_{\mathbb{R}^m} e^{-i2\pi<\tau,\nu>} C(\tau)\, d\tau .$$

Equivalently, the finite measure μ_X is the power spectral measure of $\{X(t)\}_{t\in\mathbb{R}^m}$ if and only if, for all φ and ψ belonging to $\mathrm{L}^1(\mathbb{R}^m)$

$$\operatorname{cov}\left(\int_{\mathbb{R}^m} \varphi(t) X(t)\, dt, \int_{\mathbb{R}^m} \psi(t) X(t)\, dt\right) = \int_{\mathbb{R}^m} \widetilde{\varphi}(\nu)\, \widetilde{\psi}^*(\nu)\, \mu_X(d\nu) , \tag{3.2}$$

where

$$\widetilde{\varphi}(\nu) = \int_{\mathbb{R}^m} e^{i2\pi<t,\nu>} \varphi(t)\, dt ,$$

i.e., $\widetilde{\varphi}$ is the Fourier transform of $\check{\varphi}(t) = \varphi(-t)$, $t \in \mathbb{R}^m$ (and a similar definition for $\widetilde{\psi}$).

Remark that the total mass of μ_X is the variance of the process

$$\mu_X(\mathbb{R}^m) = C_X(0) = \mathrm{Var}(X(t)) < \infty ,$$

In signal processing, the mass of the spectral measure is the *energy* of the signal.

3.1.2 Cramèr-Khinchin Representation

Recall that a centered process with orthogonal increments and base μ is a complex-valued stochastic process $\{Z(A)\}_{A\in\mathcal{B}(\mathbb{R}^m)}$ such that, for all *disjoint* measurable sets $C, D \subseteq \mathbb{R}^m$ with finite μ-measure

$$\begin{aligned}\mathrm{E}[Z(C)] &= 0\,,\\ \mathrm{E}\left[Z(C)\,Z(D)^*\right] &= 0\,,\\ \mathrm{E}\left[|Z(C)|^2\right] &= \mu(C)\ .\end{aligned}$$

Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a w.s.s. stochastic process, with Bochner spectral measure μ_X. Then, there exists a centered process $\{Z_X(A)\}_{A\subseteq\mathcal{B}(\mathbb{R}^m)}$, with orthogonal increments and base μ_X, such that

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi<\nu,t>} Z_X(d\nu) + \mathrm{E}[X]\ , \tag{3.3}$$

where the integral thereof is a Wiener integral. Note that for all functions $g \in \mathrm{L}^2_{\mathbb{C}}(\mu_X)$, the Wiener integral $\int_{\mathbb{R}^m} g(\nu)\,Z_X(d\nu)$ is well defined, and it is in $\mathrm{L}^2_{\mathbb{C}}(\mathrm{P})$; moreover

$$\mathrm{E}\left[\left|\int_{\mathbb{R}^m} g(\nu)\,Z_X(d\nu)\right|^2\right] = \int_{\mathbb{R}^m} |g(\nu)|^2\,\mu_X(d\nu)\ . \tag{3.4}$$

Equation (3.3) is called the *Cramèr-Khinchin decomposition* of the w.s.s. process $\{X(t)\}_{t\in\mathbb{R}^m}$.

3.2 Power Spectrum of Point Processes

We shall need a convenient definition of the spectrum of a stationary point process, which is not a *bona fide* w.s.s. process, and for which therefore we cannot apply as such the Bochner and Cramèr-Khinchin theorems.

3.2.1 The Covariance Measure

If N is a second order point process, the measure M_2 on $\mathbb{R}^m \times \mathbb{R}^m$, that is well and uniquely defined by

$$M_2(A\times B) = \mathrm{E}[N(A)\,N(B)]\ ,$$

is a Radon measure (indeed, if D is a bounded set of $\mathbb{R}^{2m}$, it is contained in a "square" $C \times C$, where C is a bounded measurable subset of $\mathbb{R}^m$, and therefore $M_2(D) \leq M_2(C\times C) = \mathrm{E}\left[N(C)^2\right] < \infty$).

Definition 3.1: By definition, $\mathrm{L}^2_N(M_2)$ is the collection of measurable funtions $\varphi : \mathbb{R}^m \to \mathbb{C}$ such that

$$\mathrm{E}\left[N(|\varphi|)^2\right] < \infty\,. \tag{3.5}$$

(recall the notation $N(\varphi) := \int_{\mathbb{R}^m} \varphi(t)\,N(dt)$). ◊

Condition (3.5) implies that

$$\mathrm{E}[N(|\varphi|)] < \infty\,,$$

and therefore $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\eta)$, where η is the intensity measure of the point process N (Definition 1.4).

Clearly $L^2_N(M_2)$ is a vector space that contains all bounded functions with compact support. It can be readily checked that if $\varphi, \psi \in L^2_N(M_2)$,

$$\mathrm{E}\left[N(\varphi)\, N(\psi^*)\right] = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(s)\, M_2(dt \times ds)\ . \tag{3.6}$$

Suppose that N is of second order and stationary. Then for all measurable sets $C \subseteq \mathbb{R}^m$,

$$\mathrm{E}\left[N(C+t)\right] = \mathrm{E}\left[N(C)\right]\ .$$

The intensity measure η is invariant by translation, and therefore

$$\eta(C) = \lambda\, \ell^m(C)\ ,$$

(where ℓ^m is the Lebesgue measure on $\mathbb{R}^m$) for some $\lambda \in \mathbb{R}_+$ (the intensity). In particular $L^1_{\mathbb{C}}(\eta) = L^1_{\mathbb{C}}(\mathbb{R}^m)$, and therefore from a previous remark,

$$L^2_N(M_2) \subseteq L^1_{\mathbb{C}}(\mathbb{R}^m)\,.$$

By stationarity again, for all Borel sets $A,\ B \subseteq \mathbb{R}^m$, all $t \in \mathbb{R}^m$

$$M_2\left((A+t)\times(B+t)\right) = M_2(A \times B)\ .$$

It follows from Lemma A2.7.II, p. 409 of [Daley and Vere-Jones, 2002], that there exists a Radon measure σ such that, for all $\varphi, \psi \in L^2_N(M_2)$,

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(s)\, M_2(dt \times ds) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(s+t)\, dt\right)\sigma(ds)\ . \tag{3.7}$$

Since for $\varphi, \psi \in L^1_{\mathbb{C}}(\mathbb{R}^m)$,

$$\begin{aligned}\mathrm{E}\left[N(\varphi)\right]\mathrm{E}\left[N(\psi)\right]^* &= \left(\lambda \int_{\mathbb{R}^m} \varphi(t)\, dt\right)\left(\lambda \int_{\mathbb{R}^m} \psi^*(s)\, ds\right)\\ &= \lambda^2 \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(t+s)\, dt\right) ds\,,\end{aligned}$$

we have from (3.6) and (3.7) that for $\varphi, \psi \in L^2_N(M_2)$,

$$\mathrm{cov}\left(N(\varphi), N(\psi)\right) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(t+s)\, dt\right)\Gamma_N(ds)\ ,$$

where the Radon measure

$$\Gamma_N := \sigma - \lambda^2 \ell^m \tag{3.8}$$

is called the *covariance measure* of the stationary second order point process N.

The link with the usual notion of covariance function of a *bona fide* w.s.s. signal $\{X(t)\}_{t\in\mathbb{R}^m}$ is the following. Let $C_X(\tau)$ be the covariance function of such signal. Then for all $\varphi, \psi \in L^1_{\mathbb{C}}(\mathbb{R}^m)$,

$$\mathrm{cov}\left(\int_{\mathbb{R}^m} \varphi(t)\, X(t)\, dt, \int_{\mathbb{R}^m} \psi(s)\, X(s)\, ds\right) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi(t)\, \psi^*(t+s)\, dt\right) C_X(s)\, ds.$$

3.2.2 The Bartlett Power Spectrum

The natural extension of the Bochner power spectrum (3.2) to the point process framework is the *Bartlett power spectrum* ([Bartlett, 1963], see also for instance [Neveu, 1977; Daley and Vere-Jones, 1988, 2002]).

Definition 3.2: **Bartlett spectral measure**. Let N be a simple second order stationary point process on $\mathbb{R}^m$ with intensity λ. Let B_N be a vector space of functions, here also called "test functions", such that $B_N \subseteq \mathrm{L}^2_N(M^2)$. A measure μ_N on $\mathbb{R}^m$ is called the Bartlett spectral measure of N on the domain B_N if for all $\varphi \in B_N$, the identity

$$\mathrm{Var}\left(N\left(\varphi\right)\right) = \int_{\mathbb{R}^m} \left|\widehat{\varphi}\left(\nu\right)\right|^2 \mu_N\left(d\nu\right) \tag{3.9}$$

holds, the two terms of the equality being finite. The space B_N is also called a *test function space* for the point process N. ◊

By polarization of (3.9), we have that for all $\varphi,\ \psi \in B_N$,

$$\mathrm{cov}\left(N(\varphi), N(\psi)\right) = \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_N(d\nu)\,. \tag{3.10}$$

A suitable space B_N of test functions will be determined in each situation. Ideally we want to determine the largest possible domain B_N. We are looking for conditions like, for instance, $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Note that it is necessarily contained in $\mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$ since, as we observed earlier, $\mathrm{L}^2_N(M^2) \subseteq \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$. In particular the Fourier transform of any $\varphi \in B_N$ is well-defined.

The existence and uniqueness of the Bartlett spectrum is proven in Theorem 3.2.1 below [Neveu, 1977]. The theorem also gives an interesting universal test functions space B_N.

Theorem 3.2.1 (Existence and uniqueness) *Let N be a stationary, second order point process, and let σ be the corresponding Radon measure as in (3.7). There exists a unique non-negative Radon measure $\widehat{\sigma}$ on $(\mathbb{R}^m, \mathcal{B}^m)$ such that, if φ and its Fourier transform are $O\left(1/\left|t\right|^2\right)$ as $|t| \to \infty$, then*

$$\int_{\mathbb{R}^m} \varphi\left(\nu\right)\widehat{\sigma}\left(d\nu\right) = \int_{\mathbb{R}^m} \widehat{\varphi}\left(t\right)\sigma\left(dt\right)\,, \tag{3.11}$$

and, if g satisfies the same conditions as f,

$$\mathrm{E}\left[N\left(\varphi\right)N\left(\psi^*\right)\right] = \lambda\int_{\mathbb{R}^m} \widehat{\varphi}\left(\nu\right)\widehat{\psi}^*\left(\nu\right)\widehat{\sigma}\left(d\nu\right)\,. \tag{3.12}$$

We now give some examples of the Bartlett spectra, with corresponding domain, of some point processes that are of interest in this work.

EXAMPLE 3.1: Regular grid. This example, when reduced to the univariate case, is especially interesting in the communications setup, where the point processes considered have regularly spaced points, or are obtained by jitter of such regular grids.

Consider the point process on $\mathbb{R}^2$ whose points form a regular (T_1, T_2)-grid on $\mathbb{R}^2$ with random origin, that is

$$N = \left\{\left(n_1T_1 + U_1, n_2T_1 + U_2\right), \left(n_1, n_2\right) \in \mathbb{Z}^2\right\}\,,$$

where $T_1 > 0$, $T_2 > 0$, and U_1, U_2 are independent uniform random variables on $[0, T_1]$, $[0, T_2]$ respectively. The point process is obviously second order stationary with average intensity $\lambda =$

$1/(T_1T_2)$. Its Bartlett spectral measure is

$$\mu_N = \frac{1}{T_1^2T_2^2} \sum_{(n_1,n_2)\neq(0,0)} \varepsilon_{\left(\frac{n_1}{T_1},\frac{n_2}{T_2}\right)}\,, \tag{3.13}$$

and we can take

$$B_N = \left\{ \varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^2) \text{ and } \sum_{n_1,n_2\in\mathbb{Z}} \left| \hat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) \right| < \infty \right\}. \tag{3.14}$$

Proof Conditions (3.14) guarantee that the weak Poisson formula holds true. More precisely, the left-hand side of the following equality

$$\sum_{n_1,n_2\in\mathbb{Z}} \varphi\left(u_1 + n_1T_1, u_2 + n_2T_2\right) = \frac{1}{T_1T_2} \sum_{n_1,n_2\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) e^{2i\pi\left(\frac{n_1}{T_1}u_1 + \frac{n_2}{T_2}u_2\right)} \tag{3.15}$$

is well-defined, and the equality holds for almost-all $(u_1, u_2) \in \mathbb{R}^2$, with respect to the Lebesgue measure (see for instance [Brémaud, 2002, Theorem A2.3]).

Note that the finiteness of the sum in (3.14) implies

$$\sum_{n_1,n_2\in\mathbb{Z}} \left| \hat{\varphi}\left(\frac{n_1}{T_1}u_1, \frac{n_2}{T_2}u_2\right) \right|^2 < \infty\,,$$

(in mathematical terms $\ell^1_{\mathbb{C}} \subseteq \ell^2_{\mathbb{C}}$). By (3.15)

$$N(\varphi) = \sum_{n_1,n_2\in\mathbb{Z}} \varphi\left(U_1 + n_1T_1, U_2 + n_2T_2\right) = \frac{1}{T_1T_2} \sum_{n_1,n_2\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) e^{2i\pi\left(\frac{n_1}{T_1}U_1 + \frac{n_2}{T_2}U_2\right)}\,,$$

and therefore

$$\begin{aligned}
\mathrm{E}\left[|N(\varphi)|^2\right] &= \frac{1}{T_1^2T_2^2}\mathrm{E}\left[\sum_{n_1,n_2\in\mathbb{Z}} \sum_{k_1,k_2\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) \widehat{\varphi}^*\left(\frac{k_1}{T_1}, \frac{k_2}{T_2}\right) e^{2i\pi\left(\frac{n_1-k_1}{T_1}U_1 + \frac{n_2-k_2}{T_2}U_2\right)} \right] \\
&= \frac{1}{T_1^2T_2^2} \sum_{n_1,n_2\in\mathbb{Z}} \left| \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) \right|^2.
\end{aligned}$$

Also

$$\begin{aligned}
\mathrm{E}\left[N(\varphi)\right] &= \sum_{n_1,n_2\in\mathbb{Z}} \mathrm{E}\left[\varphi\left(U_1 + n_1T_1, U_2 + n_2T_2\right)\right] \\
&= \frac{1}{T_1T_2} \sum_{n_1,n_2\in\mathbb{Z}} \int_0^{T_1}\int_0^{T_2} \varphi\left(u_1 + n_1T_1, u_2 + n_2T_2\right) du \\
&= \frac{1}{T_1T_2} \int_{\mathbb{R}^2} \varphi(t)\,dt = \widehat{\varphi}(0,0)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Var}\left(N(\varphi)\right) &= \frac{1}{T_1^2T_2^2} \sum_{n_1,n_2\in\mathbb{Z}} \left| \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) \right|^2 - \frac{1}{T_1^2T_2^2} \left|\widehat{\varphi}(0,0)\right|^2 \\
&= \frac{1}{T_1^2T_2^2} \sum_{(n_1,n_2)\neq(0,0)} \left| \widehat{\varphi}\left(\frac{n_1}{T_1} \frac{n_2}{T_2}\right) \right|^2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{\varphi}(\nu_1,\nu_2)\right|^2 \mu_N(d\nu_1 \times d\nu_2)\,,
\end{aligned}$$

where μ_N is given by (3.13). □

EXAMPLE 3.2: Cox process. Let N be a Cox point process on $\mathbb{R}^m$ with stochastic intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ (see Example 1.4). We suppose that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a w.s.s. process with mean λ and Bochner spectral measure μ_λ. Then the Bartlett spectrum of N is

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu\,, \tag{3.16}$$

on the domain $B_N = \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Moreover, in this case this is the maximal domain, since $B_N = \mathrm{L}^2_N(M_2)$.

Proof We shall denote $\mathcal{F}^\lambda_\infty$ the sigma-field generated by the stochastic process $\{\lambda(t)\}_{t\in\mathbb{R}}$. Note that if $\varphi \in \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$,

$$\mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t)|^2\,\lambda(t)dt\right] = \lambda\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt < \infty\,,$$

and therefore

$$\int_{\mathbb{R}^m} |\varphi(t)|^2\,\lambda(t)dt < \infty\,.$$

Similarly , since $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt < \infty\,.$$

By classical formulas concerning integrals with respect to Poisson processes (see for instance [Resnick, 1994]),

$$\mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,N(dt)\right)^2 |\mathcal{F}^\lambda_\infty\right] = \int_{\mathbb{R}^m} |\varphi(t)|^2\,\lambda(t)dt + \left(\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt\right)^2\,. \tag{3.17}$$

We therefore have

$$\begin{aligned}
\mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,N(dt)\right)^2\right] &= \lambda\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt + \mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt\right)^2\right]\\
&\le \lambda\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt + \left(\mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt\right]\right)^2\\
&= \lambda\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt + \left(\lambda\int_{\mathbb{R}^m} |\varphi(t)|\,dt\right)^2\,.
\end{aligned}$$

Hence, if $\varphi \in \mathrm{L}^2_N(M_2)$, then $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Also, from

$$\mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,N(dt)\right)^2\right] = \lambda\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt + \mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt\right)^2\right]\,,$$

and the fact that if $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$ then

$$\begin{aligned}
\mathrm{E}\left[\left(\int_{\mathbb{R}^m} |\varphi(t)|\,\lambda(t)dt\right)^2\right] &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |\varphi(t)|\,|\varphi(s)|\,\mathrm{E}\left[\lambda\left(t\right)\lambda\left(s\right)\right]dtds\\
&\le \mathrm{E}\left[\lambda\left(t\right)^2\right]^{1/2}\left(\int_{\mathbb{R}^m} |\varphi(t)|\,dt\right)^2 < \infty\,,
\end{aligned}$$

(this is because $\{\lambda(t)\}_{t\in\mathbb{R}}$ is w.s.s.), we see that if $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$, then $\varphi \in \mathrm{L}^2_N(M_2)$. Therefore in the case of Cox processes with a w.s.s. conditional intensity, $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ is equivalent to $\varphi \in \mathrm{L}^2_N(M_2)$.

By the conditional variance formula (see for instance [Shiryaev, 1996]),

$$\begin{aligned}
\operatorname{Var}(N(\varphi)) &= \mathrm{E}\left[\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)N(dt)\middle|\mathcal{F}_\infty^\lambda\right)\right] + \operatorname{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(t)N(dt)\middle|\mathcal{F}_\infty^\lambda\right]\right) \\
&= \mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(t)^2\lambda(t)dt\right] + \operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\lambda(t)dt\right) \\
&= \lambda\int_{\mathbb{R}^m}\varphi(t)^2dt + \operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\lambda(t)dt\right).
\end{aligned}$$

By definition of the Bochner spectrum, for $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$,

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\lambda(t)dt\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_\lambda(d\nu)\,,$$

and by the Plancherel-Parseval formula

$$\int_{\mathbb{R}^m}\varphi^2(t)dt = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,d\nu\,.$$

Therefore

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)N(dt)\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,(\mu_\lambda(d\nu)+\lambda d\nu)\,,$$

which proves that the Bartlett spectrum is indeed (3.16) on the announced domain. □

Chapter 4

The Toolbox

Summary: This chapter presents the basic tools for computing power spectra of complex signals related to random spike fields.

Our contribution: A first contribution is the fundamental isometry formula, which is the "seed" formula for the computation of the spectra of processes related to random spike fields. This general formula has a "swiss army knife" structure: by choosing the appropriate form of the formula one can derive the power spectra of a variety of spatial processes, or point processes resulting from transformations of a basic point process. We obtain known spectra as well as new ones. We give more precise characterization of the domain of application of these formulas in terms of the domain of the Bartlett spectrum of the basic point process (before transformation).

4.1 Fundamental Isometry Formula

The fundamental isometry formula is the key result for generating the toolbox that will provide the "modules" for computing the power spectrum of complex signals related to random spike fields.

Consider now a marked point process $\bar{N}$ on $\mathbb{R}^m$ with i.i.i.d. marks in the measurable space $(K, \mathcal{K})$ (see Definition 1.6). Let N be its basic point process on $\mathbb{R}^m$, assumed locally finite and simple, and let $\{Z(t)\}_{t\in\mathbb{R}^m}$ be its mark process. The family of random variables $\{Z(t)\}_{t\in\mathbb{R}^m}$ is i.i.d. with common probability distribution Q, and independent of N. Also assume that N is a second order stationary point process with Bartlett spectral measure μ_N on the domain B_N.

Let Z be a random element with distribution Q. We recall the notation: $\mathrm{L}^p_{\mathbb{C}}(\ell \times Q)$ is the set of functions $\varphi : \mathbb{R}^m \times K \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^m} \mathrm{E}\left[|\varphi(t, Z)|^p\right] \, dt < \infty \, .$$

In particular, $\varphi(t, z) \in \mathrm{L}^p_{\mathbb{C}}(Q)$ for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure).

Let $\varphi : \mathbb{R}^m \times K \to \mathbb{R}$ be a measurable function such that

$$\varphi \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \, . \tag{4.1}$$

Since $\varphi(t, z) \in \mathrm{L}^1_{\mathbb{C}}(Q)$ for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure), we can define for almost all t

$$\overline{\varphi}(t) := \mathrm{E}\left[\varphi(t, Z)\right] \, .$$

It also follows from assumption (4.1) that $\overline{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$. Therefore, $\varphi(t, z) \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$, for Q-almost all $z \in K$. Let the Fourier transforms of these two functions be denoted by $\widehat{\overline{\varphi}}$ and $\widehat{\varphi}(\cdot, z)$

respectively. Suppose moreover that

$$\varphi \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)\,. \tag{4.2}$$

Note that condition (4.2) implies that $\int_{\mathbb{R}^m} |\mathrm{E}\,[\varphi(t,Z)]|^2\, dt < \infty$, that is $\overline{\varphi} \in \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$, and for Q-almost all $z \in K$, $\varphi(\cdot, z) \in \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Observe that

$$\overline{\widehat{\varphi}}(\nu) := \mathrm{E}\,[\widehat{\varphi}(\nu, Z)] = \widehat{\overline{\varphi}}(\nu)\,.$$

Finally, suppose that

$$\overline{\varphi} \in B_N\,. \tag{4.3}$$

We can now state a *fundamental isometry formula.*

Theorem 4.1.1 (Fundamental isometry formula) *Let N and $\{Z(t)\}_{t\in\mathbb{R}^m}$ be as above, and $\varphi, \psi : \mathbb{R}^m \times K \to \mathbb{R}$ satisfy conditions (4.1), (4.2) and (4.3). Then*

$$\begin{aligned}\mathrm{cov}&\left(\int_{\mathbb{R}^m\times K} \varphi(t,z)\bar{N}\,(dt\times dz)\,, \int_{\mathbb{R}^m\times K} \psi(t,z)\bar{N}\,(dt\times dz)\right)\\ &= \int_{\mathbb{R}^m} \widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu) + \lambda \int_{\mathbb{R}^m} \mathrm{cov}\left(\widehat{\varphi}(\nu,Z), \widehat{\psi}^*(\nu,Z)\right) d\nu\,,\end{aligned} \tag{4.4}$$

where Z is a K-valued random variable with distribution Q.

Proof Formally:

$$\begin{aligned}\mathrm{E}&\left[\left(\int_{\mathbb{R}^m} \varphi(t,z)\bar{N}\,(dt\times dz)\right)\left(\int_{\mathbb{R}^m} \psi(t,z)\bar{N}\,(dt\times dz)\right)\right]\\ &= \mathrm{E}\left[\left(\sum_{t\in N}\varphi(t,Z(t))\right)\left(\sum_{t\in N}\psi(t,Z(t))\right)\right]\\ &= \mathrm{E}\left[\sum_{t,t'\in N, t\neq t'}\varphi(t,Z(t))\psi(t',Z(t'))\right] + \mathrm{E}\left[\sum_{t\in N}\varphi(t,Z(t))\psi^*(t,Z(t))\right]\\ &= \mathrm{E}\left[\sum_{t,t'\in N, t\neq t'}\overline{\varphi}(t)\overline{\psi}^*(t')\right] + \mathrm{E}\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right]\\ &= \mathrm{E}\left[\left(\sum_{t\in N}\overline{\varphi}(t)\right)\left(\sum_{t'\in N}\overline{\psi}^*(t')\right)\right] - \mathrm{E}\left[\sum_{t\in N}\overline{\varphi}(t)\overline{\psi}^*(t)\right] + \mathrm{E}\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right].\end{aligned}$$

Denote $a - b + c$ the last line. The above formal computations are justified because all the three terms are, when φ and ψ are replaced by their absolute values, finite. This follows from Schwarz's inequality, and the facts that (for a and b) $\overline{\varphi}$ and $\overline{\psi}$ are in $\mathrm{L}^2_N(M_2)$ and in $\mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$; and for c because of condition (4.2). Since

$$\mathrm{E}\left[\sum_{t\in N}\varphi(t,Z(t))\right] = \mathrm{E}\left[\sum_{t\in N}\overline{\varphi}(t)\right],$$

we have

$$\begin{aligned}\mathrm{cov}&\left(\sum_{t\in N}\varphi(t,Z(t)), \sum_{t\in N}\psi(t,Z(t))\right) =\\ &\mathrm{cov}\left(\sum_{t\in N}\overline{\varphi}(t), \sum_{t\in N}\overline{\psi}(t)\right) - \mathrm{E}\left[\sum_{t\in N}\overline{\varphi}(t)\overline{\psi}^*(t)\right] + \mathrm{E}\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right].\end{aligned}$$

Denote by A, B and C the three terms in the right-hand side of the above equation, which then reads $A - B + C$. By definition of the Bartlett spectrum, and the hypothesis (4.3),

$$A = \int_{\mathbb{R}^m} \widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu)\,.$$

By definition of the intensity λ,

$$B = \lambda \int_{\mathbb{R}^m} \overline{\varphi}(t)\overline{\psi}(t)^* dt\,, \quad C = \lambda \int_{\mathbb{R}^m} \mathrm{E}\left[\varphi(t,Z)\psi(t,Z)^*\right] dt\,.$$

By the Plancherel-Parseval identity,

$$B = \lambda \int_{\mathbb{R}^m} \widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}(\nu)^* d\nu = \lambda \int_{\mathbb{R}^m} \overline{\widehat{\varphi}}(\nu)\overline{\widehat{\psi}}(\nu)^* d\nu = \lambda \int_{\mathbb{R}^m} \mathrm{E}\left[\widehat{\varphi}(\nu,Z)\right]\mathrm{E}\left[\widehat{\psi}(\nu,Z)^*\right] d\nu\,,$$

and

$$C = \lambda \mathrm{E}\left[\int_{\mathbb{R}^m} \widehat{\varphi}(\nu,Z)\widehat{\psi}(\nu,Z)^* d\nu\right]\,,$$

and the result (4.4) follows.

□

4.2 Applications of the Fundamental Isometry Formula

We shall now consider some direct applications of the fundamental isometry formula.

4.2.1 Filtering

We then start by recalling the classical result concerning the filtering on w.s.s. signals.

Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a stochastic process and $h : \mathbb{R}^m \to \mathbb{R}$ a function. The process defined (when possible) by

$$Y(t) = \int_{\mathbb{R}^m} h(t-s)\,X(s)\,ds\,, \quad t \in \mathbb{R}^m\,, \tag{4.5}$$

corresponds to a (spatial) filtering of the process $\{X(t)\}_{t\in\mathbb{R}^m}$ with a filter with *impulse response* h.

In particular we are interested in the case where $\{X(t)\}_{t\in\mathbb{R}^m}$ is a w.s.s. process, with Bochner spectral measure μ_X. The first question that arises concerns the well definition of equation (4.5). It can be shown that for a w.s.s. process and a *stable* filtering function, *i.e.*, $h \in \mathrm{L}^1(\mathbb{R}^m)$, we have

$$\int_{\mathbb{R}^m} h(t-s)\,X(s)\,ds < \infty \quad \text{almost surely.}$$

The second question concerns the computation of the power spectrum of the filtered stochastic process and in particular the relation between the spectrum of the input and the output of the filter. We have the following corollary (see for instance [Dacunha-Castelle and Duflo, 1986]).

Corollary 4.2.1 (Filtering formula) *Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a w.s.s. process with Bochner power spectral measure μ_X. Let $h : \mathbb{R}^m \to \mathbb{C}$ be a function belonging to $\mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Then the process $\{Y(t)\}_{t\in\mathbb{R}^m}$ is w.s.s., with covariance*

$$C_Y(\tau) = \int_{\mathbb{R}^m} e^{i2\pi<\tau,\nu>} \left|\widehat{h}(\nu)\right|^2 \mu_X(d\nu) \tag{4.6}$$

and, by identification, it admits the power spectral measure

$$\mu_Y(d\nu) = \left|\widehat{h}(\nu)\right|^2 \mu_X(d\nu) . \tag{4.7}$$

We now compute the power spectrum of a shot noise with random excitation.

Corollary 4.2.2 (Shot noise with random excitation) *Let $\bar{N}$ be a marked point process (with independent i.i.d. marks) as above, and let $h : \mathbb{R}^m \times K \to \mathbb{R}$ satisfy conditions (4.3) and (4.2). Define the shot noise $\{X(t)\}_{t\in\mathbb{R}^m}$ by*

$$X(t) = \sum_{s\in N} h(t-s, Z(s)) .$$

Then

$$\mathrm{E}[X(t)] = \lambda \int_{\mathbb{R}^m} \overline{h}(t)dt ,$$

and

$$\mathrm{cov}(X(u), X(v)) = \int_{\mathbb{R}^m} e^{2i\pi<\nu,u-v>}\mu_X(d\nu) ,$$

where

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu, Z)\right]\right|^2 \mu_N(d\nu) + \lambda \mathrm{Var}\left(\widehat{h}(\nu, Z)\right) d\nu . \tag{4.8}$$

Proof It suffices to apply the fundamental isometry formula to $\varphi(t,z) = h(u-t,z)$, $\psi(t,z) = h(v-t,z)$ to obtain

$$\mathrm{cov}(X(u), X(v)) = \int_{\mathbb{R}^m} \left|\widehat{\overline{h}}(\nu)\right|^2 e^{-2i\pi<\nu,u-v>}\mu_N(d\nu) + \lambda \int_{\mathbb{R}^m} \mathrm{Var}\left(\widehat{h}(\nu, Z)\right) e^{-2i\pi<\nu,u-v>}d\nu .$$

□

We remark that when we have a deterministic filtering function, expression (4.8) becomes

$$\mu_X(d\nu) = \left|\widehat{h}(\nu)\right|^2 \mu_N(d\nu) , \tag{4.9}$$

which indeed corresponds to the filtering formula for *bona fide* w.s.s. signals if we assimilate a point process to a *bona fide* w.s.s. signal with spectral measure μ_N. The spectral measure μ_N is however *not* a finite measure as it would be for ordinary w.s.s. signals.

4.2.2 Jittering

Knowing the Bartlett spectrum μ_N of a WSS point process N, what is the Bartlett spectrum $\mu_{\widetilde{N}}$ of the point process $\widetilde{N}$ obtained by independent and identically distributed displacements of the points of N? We have the following results.

Corollary 4.2.3 (Jittered point process) *Let $\bar{N}$ be the marked point process of Theorem 4.1.1, with $K = \mathbb{R}^m$. A point process $\widetilde{N}$ is defined by*

$$\widetilde{N} = \{t + Z(t), t \in N\} .$$

Then, calling λ the intensity of N, and $\mu_{\widetilde{N}}$ the Bartlett spectrum of $\widetilde{N}$,

$$\mu_{\widetilde{N}}(d\nu) = |\phi_Z(2\pi\nu)|^2 \mu_N(d\nu) + \lambda\left(1 - |\phi_Z(2\pi\nu)|^2\right) d\nu , \tag{4.10}$$

where

$$\phi_Z(\nu) = E\left[e^{<\nu,Z>}\right] \tag{4.11}$$

is the characteristic function of the random displacements distributed as Q. We can take

$$B_{\widetilde{N}} = \left\{\widetilde{\varphi};\quad E\left[\widetilde{\varphi}(t+Z)\right] \in B_N\,,\quad \textit{and}\quad \widetilde{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)\right\}\,. \tag{4.12}$$

Proof Define $\varphi(t,z) = \widetilde{\varphi}(t+z)$. Conditions (4.1) and (4.2) for the function φ are equivalent to conditions $\tilde{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$ and $\tilde{\varphi} \in \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ respectively, since for any $p \geq 0$,

$$E\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|^p\,dt\right] = \int_{\mathbb{R}^m} |\tilde{\varphi}(t)|^p\,dt\,.$$

Condition (4.3) for the function φ is satisfied by the *ad hoc* definition of $B_{\widetilde{N}}$. We may therefore apply Theorem 4.1.1. We have

$$\widehat{\varphi}(\nu,z) = e^{2i\pi<\nu,z>}\widehat{\widetilde{\varphi}}(\nu)\,,$$
$$\widehat{\overline{\varphi}}(\nu) = \overline{\widehat{\varphi}}(\nu) = \phi_Z(\nu)\widehat{\widetilde{\varphi}}(\nu)\,.$$

Therefore

$$\mathrm{Var}\left(\widehat{\varphi}(\nu,Z)\right) = \mathrm{E}\left[\left|\widehat{\widetilde{\varphi}}(\nu)\right|^2\right] - \left|\phi_Z(\nu)\widehat{\widetilde{\varphi}}(\nu)\right|^2 = \left(1-|\phi_Z(\nu)|^2\right)\left|\widehat{\widetilde{\varphi}}(\nu)\right|^2\,.$$

Finally, applying formula (4.4),

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \widetilde{\varphi}(t)\widetilde{N}(dt)\right) = \int_{\mathbb{R}^m} \left|\widehat{\widetilde{\varphi}}(\nu)\right|^2 \mu_{\widetilde{N}}(d\nu)\,,$$

where $\mu_{\widetilde{N}}$ is as in (4.10). □

From expression (4.10) we notice that the spectrum of jittered point processes is straightforwardly obtained by "plugging in" the Bartlett spectrum of the original point process and the characteristic function of the jitter. We now provide some examples of interest, focusing on the domain of definition of the jittered spectrum.

Example 4.1: Jittered regular grid. When N is the grid process of Example 3.1, the domain of definition of the spectrum is given by

$$B_{\widetilde{N}} = \left\{\tilde{\varphi};\ \sum_{n_1,n_2\in\mathbb{Z}} \left|\hat{\tilde{\varphi}}\left(\frac{n_1}{T_1},\frac{n_2}{T_2}\right)\right| < \infty\,,\quad \text{and}\quad \tilde{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^2) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^2)\right\}\,.$$

Proof Indeed, observing that

$$\left|E\,\widehat{[\tilde{\varphi}(\cdot+Z)]}(\nu)\right| = \left|\widehat{\tilde{\varphi}}(\nu)\right|\,,$$

we see that condition $E\left[\tilde{\varphi}(t+Z)\right] \in B_N$ is equivalent to $\sum_{n_1,n_2\in\mathbb{Z}} \left|\hat{\tilde{\varphi}}(\frac{n_1}{T_1},\frac{n_2}{T_2})\right| < \infty$. □

Considering, for instance, a regular T-grid on the line and a uniform $[-\frac{T}{2},\frac{T}{2}]$ jitter, we have

$$\phi_Z\left(2\pi\nu\right) = \frac{1}{T\pi\nu}\sin\left(T\pi\nu\right)$$

and, from Example 3.1,

$$\mu_N = \frac{1}{T}\sum_{n\neq 0} \varepsilon_{\frac{n}{T}}\,.$$

Therefore

$$\mu_{\widetilde{N}}(d\nu) = \left|\frac{1}{T\pi\nu}\sin\left(T\pi\nu\right)\right|^2 \frac{1}{T}\sum_{n\neq 0}\varepsilon_{\frac{n}{T}}(d\nu) + \lambda\left(1 - \left|\frac{1}{T\pi\nu}\sin\left(T\pi\nu\right)\right|^2\right) d\nu\,.$$

EXAMPLE 4.2: Jittered Cox process. We consider the case where N is the Cox process of Example 3.2. We can take

$$B_{\widetilde{N}} = \left\{\tilde{\varphi}\,;\, \tilde{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)\right\}\,.$$

Indeed condition $E\left[\tilde{\varphi}(t+Z)\right] \in B_N$, that is, in this particular case, $E\left[\tilde{\varphi}(t+Z)\right] \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$, is exactly $\tilde{\varphi} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$.

In the case of a shot noise, the question of the effect of jittering on its power spectrum has an even more interesting impact for applications. We have the following result.

Corollary 4.2.4 (Jittered shot noise) *Consider a shot noise $\{X\left(t\right)\}_{t\in\mathbb{R}^m}$ admitting a power spectral measure μ_X. Suppose now that the underlying point process is affected by i.i.d. random displacements of its points and assume the setup of Corollary 4.2.3. Denote with $\{\widetilde{X}\left(t\right)\}_{t\in\mathbb{R}^m}$ the resulting shot noise process. Then, its power spectral measure is given by*

$$\mu_{\widetilde{X}}(d\nu) = \left|\widehat{h}(\nu)\right|^2 \left|\phi_Z\left(\nu\right)\right|^2 \mu_N(d\nu) + \lambda\left|\widehat{h}(\nu)\right|^2 \left(1 - \left|\phi_Z\left(\nu\right)\right|^2\right) d\nu\,.$$

In terms of the power spectral measure μ_X of the original shot noise $\{X\left(t\right)\}_{t\in\mathbb{R}^m}$, the previous expression reads

$$\mu_{\widetilde{X}}(d\nu) = \left|\phi_Z\left(\nu\right)\right|^2 \mu_X(d\nu) + \lambda\left|\widehat{h}(\nu)\right|^2 \left(1 - \left|\phi_Z\left(\nu\right)\right|^2\right) d\nu\,. \tag{4.13}$$

Proof It suffices to apply the fundamental isometry formula to $\varphi\left(t,z\right) = h\left(u-t-z\right)$ and $\psi\left(t,z\right) = h\left(v-t-z\right)$. □

The above formula is of great importance: it explicitly shows how the jitter affects the power spectrum of the original shot noise, easily allowing to take into account such a random effect.

4.2.3 Thinning

As for the case of jittering, we investigate the impact that i.i.d. losses of the points of N have on the power spectrum. We have the following result.

Corollary 4.2.5 (Thinned point process) *Consider the marked point process of Theorem 4.1.1 (fundamental isometry formula), with $K = \{0,1\}$ and basic point process N admitting a Bartlett spectrum μ_N with domain B_N. We define the thinned point process $\widetilde{N}$ as*

$$\widetilde{N} = \{t, t \in N \mid Z(t) = 1\}\,.$$

Then, calling λ the intensity of N, and $\mu_{\widetilde{N}}$ the Bartlett spectrum of $\widetilde{N}$,

$$\mu_{\widetilde{N}}(d\nu) = \left|\mathrm{E}\left[Z\right]\right|^2 \mu_N(d\nu) + \lambda \mathrm{Var}\left(Z\right) d\nu\,. \tag{4.14}$$

Now $B_{\widetilde{N}} = B_N$

Proof The proof follows the same path as the proof of Corollary 4.2.3, where now $\varphi(t,z) = z\varphi(t)$, $\widehat{\overline{\varphi}}(\nu) = \mathrm{E}[Z]\,\widehat{\varphi}(t)$ and $\mathrm{cov}(\widehat{\varphi}(\nu,Z), \widehat{\varphi}^*(\nu,Z)) = |\widehat{\varphi}(\nu)|^2 \,\mathrm{Var}(Z)$. □

Concerning the effect of random losses of the points of N on the spectrum of its filtered version, we have the following result.

Corollary 4.2.6 (Thinned shot noise) *Consider a shot noise $\{X(t)\}_{t\in\mathbb{R}^m}$ admitting a power spectral measure μ_X. Suppose now that the underlying point process is affected by i.i.d. random losses of its points and assume the setup of Corollary 4.2.5. Denote with $\{\widetilde{X}(t)\}_{t\in\mathbb{R}^m}$ the resulting shot noise process. Then, its power spectral measure is given by*

$$\mu_{\widetilde{X}}(d\nu) = |\mathrm{E}[Z]|^2\,\mu_X(d\nu) + \lambda \left|\widehat{h}(\nu)\right|^2 \mathrm{Var}(Z)\,d\nu\,. \tag{4.15}$$

Proof It suffices to apply the fundamental isometry formula to $\varphi(t,z) = zh(u-t)$ and $\psi(t,z) = zh(v-t)$. □

The above formula is of great importance: it explicitly shows how the thinning affects the power spectrum of the original shot noise and they easily allows to take into account such random effects. Moreover, we remark that the spectrum formula also applies in the case of a shot noise with i.i.d. random amplitudes: it is sufficient to choose as K the set of the possible amplitude values.

4.2.4 Clustering

We now consider the power spectrum of the cluster point process defined in Section 2.4.

Corollary 4.2.7 *Recall the definition of the two cluster point processes (equations (2.4) and (2.5)*

$$N_{\mathrm{c}}(C) = \sum_{t\in N} Z(t, C-t)\,,$$
$$N'_{\mathrm{c}}(C) = N(C) + \sum_{t\in N} Z(t, C-t)\,.$$

where the clusters $\{Z(t)\}_{t\in\mathbb{R}^m}$ form an i.i.d. collection of point processes on $\mathbb{R}^m$, independent of N, with the notation $Z(t)(C) = Z(t,C)$. We assume that the seed, i.e., the spatial point process N, is stationary with intensity $\lambda > 0$, Bartlett spectral measure μ_N, and set of test functions B_N. Let Z be a point process on $\mathbb{R}^m$ with the same distribution as the common distribution of the $Z(t)$'s, assume

$$\mathrm{E}[Z(\mathbb{R}^m)] < \infty\,,$$

and define

$$\psi_Z(\nu) = \mathrm{E}\left[\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,t\rangle} Z(dt)\right]\,.$$

Then, we have

$$\mu_{N_{\mathrm{c}}}(d\nu) = |\psi_Z(\nu)|^2\,\mu_N(d\nu) + \lambda \mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,s\rangle} Z(ds)\right) d\nu\,. \tag{4.16}$$

and

$$\mu_{N'_{\mathrm{c}}}(d\nu) = |1+\psi_Z(\nu)|^2\,\mu_N(d\nu) + \lambda \mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,s\rangle} Z(ds)\right) d\nu. \tag{4.17}$$

Proof First of all remark that existence and finiteness of $\psi_Z(\nu)$ is guaranteed by the condition

$$\mathrm{E}\left[Z\left(\mathbb{R}^m\right)\right] < \infty.$$

In particular, Z is almost surely a finite point process. We would like to compute the Bartlett spectrum of N_c and N'_c. We start with N'_c. Formally

$$\mathrm{Var}\left(\sum_{t\in N_\mathrm{c}}\varphi(t)\right) = \mathrm{Var}\left(\sum_{t\in N}\left\{\varphi(t) + \int_{\mathbb{R}^m}\varphi(t+s)\,Z(t,ds)\right\}\right) = \mathrm{Var}\left(\sum_{t\in N}\varphi(t,Z(t))\right),$$

where

$$\varphi(t,z) = \varphi(t) + \int_{\mathbb{R}^m}\varphi(t+s)\,z(ds)\;.$$

Formally

$$\mathrm{E}\left[\varphi(t,z)\right] = \varphi(t) + \mathrm{E}\left[\int_{\mathbb{R}^m}\varphi(t+s)\,z(ds)\right]$$

$$\begin{aligned}\widehat{\varphi}(\nu,z) &= \widehat{\varphi}(\nu) + \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t+s)\,z(ds)\right)e^{-2i\pi\langle\nu,t\rangle}dt\\ &= \widehat{\varphi}(\nu) + \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t+s)\,e^{-2i\pi\langle\nu,t\rangle}dt\right)z(ds)\\ &= \widehat{\varphi}(\nu) + \int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\,e^{2i\pi\langle\nu,s\rangle}z(ds)\\ &= \widehat{\varphi}(\nu)\left(1 + \int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}z(ds)\right)\end{aligned}$$

Note that the exchange of order of integration is not a problem if z is a finite point process, in particular if z is replaced by its random version Z. Note also that $\varphi\in B_{N_\mathrm{c}}$ (not yet identified) is necessarily in $\mathrm{L}^1_\mathbb{C}(\mathbb{R}^m)$, since $B_{N_\mathrm{c}}\subseteq \mathrm{L}^1_\mathbb{C}(\mathbb{R}^m)$.

Also

$$\widehat{\widehat{\varphi}}(\nu) = \widehat{\varphi}(\nu)\left(1+\psi_Z(\nu)\right)$$

Applying formally Theorem 4.1.1, we obtain

$$\begin{aligned}\mathrm{Var}\left(\sum_{t\in N}\varphi(t,Z(t))\right) &= \int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2\left|1+\psi_Z(\nu)\right|^2\mu_N(d\nu)\\ &\quad + \lambda\int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2\mathrm{Var}\left(1+\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}Z(ds)\right)d\nu.\end{aligned}$$

Observe that

$$\mathrm{Var}\left(1+\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}Z(ds)\right) = \mathrm{Var}\left(\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}Z(ds)\right)$$

to obtain

$$\mathrm{Var}\left(\sum_{t\in N_\mathrm{c}}\varphi(t)\right) = \int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2\mu_{N_\mathrm{c}}(d\nu)\;,$$

where

$$\mu_{N'_\mathrm{c}}(d\nu) = \left|1+\psi_Z(\nu)\right|^2\mu_N(d\nu) + \lambda\mathrm{Var}\left(\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}Z(ds)\right)d\nu.$$

is the Bartlett spectrum of N_c. Similar computations lead to

$$\mu_{N_\mathrm{c}}(d\nu) = \left|\psi_Z(\nu)\right|^2\mu_N(d\nu) + \lambda\mathrm{Var}\left(\int_{\mathbb{R}^m}e^{2i\pi\langle\nu,s\rangle}Z(ds)\right)d\nu\,.$$

To obtain the corresponding domains B_{N_c} and $B_{N'_\mathrm{c}}$, it suffices to ask that the condition for $\varphi(t,z)$ in Theorem 4.1.1 (fundamental isometry formula) are satisfied. □

An interesting example is given by the spectrum of a renewal cluster point processes. Such a spectrum expression plays an important role in network traffic analysis [Hohn et al., 2003; Hohn and Veitch, 2003].

EXAMPLE 4.3: Renewal clusters. Consider the renewal cluster point process (2.3) of Example 2.7, and notice that, as remarked at the end of Section 2.4, it corresponds to the cluster point process of equation (2.5), *i.e.*,

$$N'_{\mathrm{c}} = N + \sum_{n\in\mathbb{Z}} \sum_{l=1}^{L_n} \varepsilon_{T_n + t_l^{(n)}} .$$

We then consider the spectrum expression given by (4.17), where we have now

$$\int_{\mathbb{R}^m} e^{2i\pi\langle \nu, s\rangle} Z\left(ds\right) = \int_{\mathbb{R}} e^{2i\pi\nu s} \sum_{j=1}^{L} \varepsilon_{t_j}\left(ds\right) = \sum_{j=1}^{L} e^{2i\pi\nu t_j}$$

Let S be a random variable with the same distribution as the common distribution of the i.i.d. inter-arrival times $S_l^{(n)} = t_l^{(n)} - t_{l-1}^{(n)}$, $l \geq 1$, $n \in \mathbb{Z}$. Call ϕ_S the characteristic function of the random variable S and g_L the generating function of the random variable L. Define

$$\psi_Z^{|2|}\left(\nu\right) = \mathrm{E}\left[\left|\int_{\mathbb{R}^m} e^{2i\pi\langle \nu, s\rangle} Z\left(ds\right)\right|^2\right] .$$

We have (see the proof below)

$$\psi_Z\left(\nu\right) = \begin{cases} \frac{\phi_S(2i\pi\nu)}{1-\phi_S(2i\pi\nu)}\left(1 - \phi_S\left(2i\pi\nu\right) g_L\left(\phi_S\left(2i\pi\nu\right)\right)\right) , & \nu \neq 0 ; \\ \mathrm{E}\left[L\right] , & \nu = 0 , \end{cases}$$

$$\psi_Z^{|2|}\left(\nu\right) = 2\mathrm{Re}\left(\frac{\phi_S\left(2\pi\nu\right)}{\left(\phi_S\left(2\pi\nu\right) - 1\right)^2}\left(g_L\left(\phi_S\left(2\pi\nu\right)\right) - 1\right)\right) - 2\mathrm{Re}\left(\frac{1}{\left(\phi_S\left(2\pi\nu\right) - 1\right)}\mathrm{E}\left[L\right]\right) - \mathrm{E}\left[L\right]$$

for $\nu \neq 0$, and $\psi_Z^{|2|}\left(0\right) = \mathrm{E}\left[L^2\right]$. We remark that

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle \nu, s\rangle} Z\left(ds\right)\right) = \psi_Z^{|2|}\left(\nu\right) - \left|\psi_Z\left(\nu\right)\right|^2 .$$

Therefore the expressions of $\psi_Z\left(\nu\right)$ and $\psi_Z^{|2|}\left(\nu\right)$ straightforwardly give the power spectral measure (4.17) for a renewal cluster point process.
Proof Here

$$\psi_Z\left(\nu\right) = \sum_{k\in\mathbb{N}} \sum_{j=1}^{k} \left(\mathrm{E}\left[e^{2i\pi\nu S}\right]\right)^j \mathrm{P}\left(L = k\right) .$$

Therefore, for $\nu \neq 0$ we have

$$\begin{aligned} \psi_Z\left(\nu\right) &= \sum_{k\in\mathbb{N}} \sum_{j=0}^{k} \phi_S\left(2i\pi\nu\right)^j \mathrm{P}\left(P = k\right) \\ &= \sum_{k\in\mathbb{N}} \frac{1 - \phi_S\left(2i\pi\nu\right)^{k+1}}{1 - \phi_S\left(2i\pi\nu\right)} \mathrm{P}\left(P = k\right) \\ &= \frac{1}{1 - \phi_S\left(2i\pi\nu\right)}\left(1 - \phi_S\left(2i\pi\nu\right) g_P\left(\phi_S\left(2i\pi\nu\right)\right)\right) \end{aligned}$$

where ϕ_S is the characteristic function of S and g_L the generating function of L. Finally

$$\psi_Z(\nu) = \begin{cases} \frac{1}{1-\phi_S(2i\pi\nu)}\left(1-\phi_S(2i\pi\nu)\, g_L(\phi_S(2i\pi\nu))\right) & \text{for } \nu \neq 0\,; \\ 1+\mathrm{E}[L] & \text{for } \nu = 0\,. \end{cases}$$

Concerning the second order moment, call

$$\psi_Z^{|2|}(\nu) = \mathrm{E}\left[\left|\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,s\rangle} Z(ds)\right|^2\right].$$

We have

$$\psi_Z^{|2|}(\nu) = \mathrm{E}\left[\sum_{j=0}^{L}\sum_{l=0}^{L} e^{2i\pi\nu\left(t_l-t_j\right)}\right]$$

Therefore for $\nu = 0$

$$\psi_Z^{|2|}(\nu) = 1 + 2\mathrm{E}[L] + \mathrm{E}\left[L^2\right],$$

while for $\nu \neq 0$

$$\begin{aligned} \psi_Z^{|2|}(\nu) &= 1+\mathrm{E}[L]+2\mathrm{Re}\left(\mathrm{E}\left[\sum_{0\le j<l\le L} e^{2i\pi\nu\left(t_l-t_j\right)}\right]\right) \\ &= 1+\mathrm{E}[L]+2\mathrm{Re}\left(\sum_{k=1}^{\infty}\sum_{0\le j<l\le k}\mathrm{E}\left[e^{2i\pi\nu\left(t_l-t_j\right)}\right]\mathrm{P}(L=k)\right) \end{aligned}$$

Now, for $l \geq j$

$$\mathrm{E}\left[e^{2i\pi\nu\left(t_l-t_j\right)}\right] = \mathrm{E}\left[e^{2i\pi\nu\left(S_{j+1}+\ldots+S_l\right)}\right] = \phi_S(2\pi\nu)^{l-j}$$

Therefore

$$\psi_Z^2(\nu) = 1+\mathrm{E}[L]+2\mathrm{Re}\left(\sum_{k=1}^{\infty}\mathrm{P}(L=k)\sum_{0\le j<l\le k}\phi_S(2\pi\nu)^{l-j}\right)$$

After rearranging and simplification we obtain

$$\psi_Z^{|2|}(\nu) \;=\; 1 \;+\; \mathrm{E}[L] \;+\; 2\mathrm{Re}\left\{\frac{\phi_S(2\pi\nu)}{\left(\phi_S(2\pi\nu)-1\right)^2}\left[\phi_S(2\pi\nu)\left(g_L(\phi_S(2\pi\nu))-\mathrm{E}[L]-1\right)+\mathrm{E}[L]\right]\right\}$$

□

A very particular case is when the seeds form a Poisson point process, with intensity λ, and they are not systematically counted as points of the cluster point process, *i.e.*,

$$N_c = \sum_{n\in\mathbb{Z}}\sum_{l=1}^{L_n}\varepsilon_{T_n+t_l^{(n)}}$$

We then use formula (4.16), with $\mu_N = \lambda d\nu$, obtaining

$$\mu_{N_c}(d\nu) = \lambda\psi_Z^{|2|}(\nu)\, d\nu\,. \tag{4.18}$$

Chapter 5

Modulated Spikes

Summary: We present some tools for the computation of spectra related to modulated spike fields. They can be considered as an extension of the "toolbox" we have presented in Chapter 4.

Our contribution: Concerning the spectrum of a modulated spike field, we give details for the informal proof of Daley and Vere-Jones [1988, 2002], details are are useful to determine the class of test functions for which the defining formula of the spectrum is true. We also present an extension of the fundamental isometry formula that applies to modulated spike fields.

5.1 Extended Bochner Spectrum

We consider a modulated random spike field, as described in Section 2.5. In particular we assume in the following that

- the basic point process N is simple, second-order, and stationary, and that it admits a Bartlett spectrum μ_N with a vector space of test functions B_N;
- the modulating process $\{V(t)\}_{t\in\mathbb{R}^m}$ is w.s.s. process with Bochner power spectrum μ_V.

Recall that a modulated point process can be seen as the generalized process

$$Y(t) = \sum_{s\in N} V(s)\,\delta(t-s)\ , \quad t \in \mathbb{R}^m\,,$$

(see Section 2.5). We define the extended Bochner spectrum of a modulated spike field to be a Radon measure $\mu_Y(d\nu)$ such that, for any $\varphi(t) \in B_Y$,

$$\operatorname{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\,V(t)\,N(dt)\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2\,\mu_Y(d\nu)\ , \tag{5.1}$$

where B_Y is a large enough vector space of functions, here also called the "test functions". By "large enough", we mean that there cannot be two different Radon measures μ_Y verifying (5.1) for all $\varphi \in B_Y$. Observe that, since, formally,

$$\begin{aligned}\int_{\mathbb{R}^m} \varphi(t)\,Y(t)\,dt &= \int_{\mathbb{R}^m} \varphi(t)\left(\sum_{s\in N} V(s)\,\delta(t-s)\right)dt \\ &= \sum_{s\in N} \varphi(s)\,V(s) \\ &= \int_{\mathbb{R}^m} \varphi(t)\,V(t)\,N(dt)\ ,\end{aligned}$$

equality (5.1) becomes, formally,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\, Y(t)\, dt\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) \,, \tag{5.2}$$

making the connection with the usual notion of Bochner spectrum.

We present three results regarding the spectrum of a modulated spike field. They respectively concern the following modulating processes:

- a continuous time process, independent of the point process;
- a continuous time process dependent on the point process;
- a time series, independent of the point process.

The first result is the following.

Theorem 5.1.1 (Extended Bochner - independent case) *Suppose that the stochastic process $\{V(t)\}$ is independent from the point process N. Then, the generalized process*

$$Y(t) = \sum_{s \in N} V(s)\, \delta(t-s)$$

admits the extended Bochner power spectral measure

$$\mu_Y = \mu_N * \mu_V + \lambda^2 \mu_V + |\mathrm{E}[V]|^2 \mu_N \,. \tag{5.3}$$

If B_N is stable with respect to multiplications by complex exponential functions, we can take $B_Y = B_N$.

Proof Using the Cramèr-Khinchin decomposition (3.3), we have

$$\begin{aligned}\int_{\mathbb{R}^m} \varphi(t)\, V(t)\, N(dt) &= \int_{\mathbb{R}^m} \varphi(t) \left(\int_{\mathbb{R}^m} e^{2i\pi<\nu,t>}\right) Z_V(d\nu) + \mathrm{E}[V] \int_{\mathbb{R}^m} \varphi(t)\, N(dt) \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt)\right) Z_V(d\nu) + \mathrm{E}[V] \int_{\mathbb{R}^m} \varphi(t)\, N(dt)\,,\end{aligned}$$

where we have formally exchanged the order of integration. Since the integrals with respect to $N(dt)$ and with respect to $Z_V(d\nu)$ are of a different nature (one is a usual infinite sum, the other is a Wiener integral), this exchange must be formally justified, which we do after the proof. Using the conditional variance formula (see for instance [Shiryaev, 1996]), we have

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\, V(t)\, N(dt)\right) \\ &\quad = \underbrace{\mathrm{E}\left[\mathrm{Var}\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt)\right) Z_V(d\nu) + \mathrm{E}[V]\, N(\varphi) \middle| \mathcal{F}_\infty^N\right)\right]}_{\alpha} \\ &\qquad + \underbrace{\mathrm{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt)\right) Z_V(d\nu) + \mathrm{E}[V]\, N(\varphi) \middle| \mathcal{F}_\infty^N\right]\right)}_{\beta}\end{aligned}$$

Observe that, since $\varphi \in \mathrm{L}^2(M_2)$,

$$\left|\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt)\right|^2 \leq \left|\int_{\mathbb{R}^m} |\varphi(t)|\, N(dt)\right|^2 < \infty\,, \text{ P-a.s.} \tag{5.4}$$

Using the fact that, when N is fixed, $\mathrm{E}[V]\int_{\mathbb{R}^m}\varphi(t)\,N(dt)$ is deterministic,

$$\alpha = E\left[\mathrm{Var}\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_V(d\nu)\middle|\,\mathcal{F}_\infty^N\right)\right]$$

by eq. (3.4) and (5.4)

$$\begin{aligned}
&= E\left[\int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\mu_V(d\nu)\right]\\
&= \int_{\mathbb{R}^m}E\left[\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\right]\mu_V(d\nu)\\
&= \int_{\mathbb{R}^m}\left(\mathrm{Var}\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)+\left|\mathrm{E}\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\right)\mu_V(d\nu)
\end{aligned}$$

by the hypothesis on B_N

$$\begin{aligned}
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx)+\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}\lambda dt\right|^2\right)\mu_V(d\nu)\\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx)\right)\mu_V(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(-\nu)|^2\mu_V(d\nu)\\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x+\nu)|^2\mu_N(dx)\right)\mu_V(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(+\nu)|^2\mu_V(d\nu)\\
&= \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2(\mu_N*\mu_V)(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_V(d\nu)
\end{aligned}$$

and, since

$$E\left[\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_V(d\nu)\middle|\,\mathcal{F}_\infty^N\right]=0,$$

we have

$$\beta = \mathrm{Var}\left(\mathrm{E}[V]\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right)=|\mathrm{E}[V]|^2\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_N(d\nu)\ \ (\text{since }\varphi\in B_N).$$

Finally,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,Y(t)\,dt\right)=\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\left(\mu_N*\mu_V+\lambda^2\mu_V+|\mathrm{E}[V]|^2\mu_N\right)(d\nu),$$

that is, $\{Y(t)\}_{t\in\mathbb{R}^m}$ admits an extended Bochner spectral measure given by equation (5.3). □

It now remains to validate the exchange of integrals perpetrated at the begining of the previous proof.

Lemma 5.1.1 *Let N be a simple locally bounded stationary point process defined on $\mathbb{R}^m$ and admitting a Bartlett spectrum μ_N. Let M_2 be its second moment measure. Let $\{V(t)\}_{t\in\mathbb{R}^m}$ be a w.s.s. process with Cramèr-Khinchin decomposition Z_V and power spectral measure μ_V. Then, for all $\varphi\in\mathrm{L}^2(M_2)$*

$$\int_{\mathbb{R}^m}\varphi(t)\,V(t)\,N(dt)=\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_V(d\nu).\qquad(5.5)$$

Proof We do the proof in the univariate case. The multivariate case follows the same lines, with more notation. The left-hand side of (5.5) is

$$A = \sum_{n\in\mathbb{Z}} \varphi(T_n) V(T_n) = \lim_{c\uparrow\infty} \sum_{n\in\mathbb{Z}} \varphi(T_n) V(T_n) 1_{[-c,+c]}(T_n) = \lim_{c\uparrow\infty} A(c)$$

where the limit is in $\mathrm{L}^1(P)$. Indeed

$$\begin{aligned} E[|A - A(c)|] &\le E\left[\int_{\overline{[-c,+c]}} |\varphi(t) V(t)| N(dt)\right] \\ &= \int_{\overline{[-c,+c]}} |\varphi(t)| E[|V(t)|] \lambda dt \le \lambda K \int_{\overline{[-c,+c]}} |\varphi(t)| dt \end{aligned}$$

where,by Schwarz's inequality $E[|V(t)|] \le E\left[|V(t)|^2\right]^{\frac{1}{2}} = E\left[|V(0)|^2\right]^{\frac{1}{2}}$,

$$K = \sup_t E[|V(t)|] < \infty$$

Therefore, since $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R})$, $\lim_{c\uparrow\infty} \mathrm{E}[|A - A(c)|] = 0$.

The right-hand side is

$$B = \lim_{c\uparrow\infty} \int_{\mathbb{R}} \left(\int_{[-c,+c]} \varphi(t) e^{2i\pi<\nu,t>} N(dt)\right) Z_V(d\nu) = \lim_{c\uparrow\infty} B(c)$$

where the limit is in $\mathrm{L}^2(P)$. Indeed

$$\begin{aligned} E\left[|B - B(c)|^2\right] &= E\left[\left|\int_{\mathbb{R}} \left(\int_{\overline{[-c,+c]}} \varphi(t) e^{2i\pi\nu t} N(dt)\right) Z_V(d\nu)\right|^2\right] \\ &= E\left[E\left[\left|\int_{\mathbb{R}} \left(\int_{\overline{[-c,+c]}} \varphi(t) e^{2i\pi\nu t} N(dt)\right) Z_V(d\nu)\right|^2 \middle| \mathcal{F}^N_\infty\right]\right] \\ &= E\left[\int_{\mathbb{R}} \left|\int_{\overline{[-c,+c]}} \varphi(t) e^{2i\pi\nu t} N(dt)\right|^2 \mu_V(d\nu)\right]. \end{aligned}$$

Denote $\varphi_c(t) = \varphi(t) 1_{\overline{[-c,c]}}(t)$. Then

$$\mathrm{E}\left[\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \varphi_c(t) e^{2i\pi\nu t} N(dt)\right|^2 \mu_V(d\nu)\right] = \int_{\mathbb{R}} \mathrm{E}\left[\left|\int_{\mathbb{R}} \varphi_c(t) e^{2i\pi\nu t} N(dt)\right|^2\right] \mu_V(d\nu).$$

But

$$\begin{aligned} \mathrm{E}\left[\left|\int_{\mathbb{R}} \varphi_c(t) e^{2i\pi\nu t} N(dt)\right|^2\right] &\le \mathrm{E}\left[\left(\int_{\mathbb{R}} |\varphi_c(t)| N(dt)\right)^2\right] \\ &= \int_{\mathbb{R}\times\mathbb{R}} |\varphi_c(t)| |\varphi_c(s)| M_2(dt \times ds), \end{aligned}$$

a quantity that tends to 0 as $c \uparrow \infty$, by dominated convergence. Dominated convergence applied to the *finite* measure μ_V then yields the desired L^2 convergence.

But

$$\begin{aligned}A(c) &= \sum_{n\in\mathbb{Z}} \varphi(T_n) V(T_n) 1_{[-c,+c]} \\ &= \sum_{n\in\mathbb{Z}} \varphi(T_n) \left(\int_{\mathbb{R}} e^{2i\pi\nu T_n} Z_V(d\nu)\right) 1_{[-c,+c]}(T_n) \\ &= \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} \varphi(T_n) e^{2i\pi\nu T_n} 1_{[-c,+c]}(T_n)\right) Z_V(d\nu) \\ &= B(c),\end{aligned}$$

where we have used the fact that the sums involved are finite. Thus

$$\lim_{c\uparrow\infty} A(c) = \begin{cases} A & \text{in } \mathrm{L}^1 \\ B & \text{in } \mathrm{L}^2 \end{cases}$$

from which it follows that $A = B$, a.s. (use the fact that if a sequence of random variables converges in L^1 or L^2 to some random variable, one can extract a subsequence that converges a.s. to the same random variable). □

The second result concerns the situation where the intensity of the point process, *i.e.*, the *sampling rate*, depends on the modulating process V. More precisely, the model for the point process N, *i.e.*, *the sampler*, is now a Cox process on $\mathbb{R}^m$, with the conditional (w.r.t. V) intensity of the form

$$\lambda(t) = \lambda(t, V).$$

For instance, in the univariate case, $\lambda(t) = |V(t)|^2$, $\lambda(t) = \left|\dot{V}(t)\right|^2$ where $\dot{V}$ is the derivative at t of $t \to V(t)$. More complicated functionals can be considered.

We have the following result.

Theorem 5.1.2 (Extended Bochner - dependent intensity) *Assume that*

$$\mathrm{E}\left[V(t)^2 \lambda(t, V)^2\right] < \infty, \quad \forall t \in \mathbb{R}^m,$$

and that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ *is a locally integrable process. Let* μ_Z *be the power spectrum of the stationary process*

$$Z(t) = V(t)\lambda(t).$$

Then,

$$\mu_Y(d\nu) = \mu_Z(d\nu) + \overline{V^2\lambda} d\nu \tag{5.6}$$

where we have denoted $\overline{V^2\lambda} = \mathrm{E}\left[V(t)^2 \lambda(t)\right]$ *(independent of t).*

Proof In order to compute the Bartlett spectrum of $Y(t)$, we have, as in the independent case, to evaluate the variance of

$$\int_{\mathbb{R}^m} \varphi(t) Y(t) dt = \int_{\mathbb{R}^m} \varphi(t) V(t) N(dt)$$

for all $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$. It holds that

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) V(t) N(dt)\right) = \\ &\quad \mathrm{E}\left[\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) V(t) N(dt) \Big| \mathcal{F}_\infty^V\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t) V(t) N(dt) \Big| \mathcal{F}_\infty^V\right]\right) \\ &\quad = \mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t)|^2 |V(t)|^2 \lambda(t, V) dt\right] + \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) V(t) \lambda(t, V) dt\right).\end{aligned}$$

By definition of μ_Z, we have

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) V(u) \lambda(u)\, du\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu).$$

Therefore, recalling the notation $\mathrm{E}\left[V(t)^2 \lambda(t)\right] = \overline{V^2\lambda}$ (independent of t), we have

$$\begin{aligned}\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) V(t) N(dt)\right) &= \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) V(u) \lambda(u)\, du\right) + \overline{V^2\lambda}\int_{\mathbb{R}^m} |\varphi(t)|^2\, dt \\ &= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu) + \overline{V^2\lambda}\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2\, d\nu \\ &= \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\left(\mu_Z(d\nu) + \overline{V^2\lambda} d\nu\right)\end{aligned}$$

and result (5.6) follows. □

The third result concerns the situation where the point process N is a renewal one, and it is modulated by a w.s.s. time series $\{A_n\}_{n\in\mathbb{Z}}$, independent from N. Remark that, for a general correlated time series, the fundamental isometry formula does not apply since we cannot model the correlated time series with i.i.d. marks. We call $R_A(k) = \mathrm{E}[A_{n+k}A_n^*]$ the correlation function of the time series.

Theorem 5.1.3 (Extended Bochner - time series) *Assume that the time series $\{A_n\}_{n\in\mathbb{Z}}$ and the renewal point process N are independent. Suppose moreover that the latter has inter-arrival times $\{S_n\}_{n\in\mathbb{Z}}$ and a renewal function $U(t)$ that admits a density, i.e.*

$$U(t) = \int_0^t u(s)\, ds,$$

such that

$$\int_0^\infty |u(t) - \lambda|\, dt < \infty \tag{5.7}$$

Suppose moreover that the time series has a summable covariance function

$$\sum_{k\in\mathbb{Z}} \left| R_A(k) - |\mathrm{E}[A]|^2 \right| < \infty. \tag{5.8}$$

Then, the power spectral measure of the point process N modulated by the time series $\{A_n\}_{n\in\mathbb{Z}}$, associated to the domain B_Y, is given by

$$\mu_Y(d\nu) = \lambda\left(2\mathrm{Re}\left\{\sum_{k\geq 0} \phi_S^k(2\pi\nu) R_A(k)\right\} - R_A(0)\right) d\nu - \lambda^2 \mathrm{E}[A]^2 \varepsilon_0(d\nu), \tag{5.9}$$

where $\phi_S(u) = \mathrm{E}\left[e^{iuS}\right]$ and ε_a is the Dirac measure centered in a.

Proof Without loss of generality, we do the proof for a renewal process on the half line.

$$\begin{aligned}\mathrm{cov}\Big(\sum_{n\geq 1} A_n\varphi(T_n), \sum_{n\geq 1} A_n\psi(T_n)\Big) =& \mathrm{E}\Big[\sum_{n\geq 1} A_n\varphi(T_n) \sum_{m\geq 1} A_m\psi(T_m)^*\Big] \\ &- \mathrm{E}\Big[\sum_{n\geq 1} A_n\varphi(T_n)\Big]\mathrm{E}\Big[\sum_{m\geq 1} A_m\psi(T_m)^*\Big]\end{aligned}$$

We have

$$\mathrm{E}\Big[\Big(\sum_{n\geq 1} A_n\varphi(T_n)\Big)\Big(\sum_{m\geq 1} A_m^*\psi(T_m)\Big)^*\Big] = \underbrace{\mathrm{E}\Big[\sum_{n>m\geq 1} A_n\varphi(T_n)A_m^*\psi^*(T_m)\Big]}_{\alpha} + \underbrace{\mathrm{E}\Big[\sum_{m>n\geq 1} A_n\varphi(T_n)A_m^*\psi^*(T_m)\Big]}_{\beta} + \underbrace{\mathrm{E}\Big[\sum_{n\geq 1} |A_n|^2\varphi(T_n)\psi^*(T_n)\Big]}_{\gamma}$$

$$\begin{aligned}\alpha =& \mathrm{E}\left[\sum_{n>m\geq 1} A_nA_m^*\varphi(T_m+S_{m+1}+\ldots+S_n)\psi^*(T_m)\right]\\ (\text{Fubini}) =& \sum_{n>m\geq 1}\mathrm{E}[A_nA_m^*]\,\mathrm{E}[\varphi(T_m+S_{m+1}+\ldots+S_n)\psi^*(T_m)]\\ =& \sum_{n>m\geq 1}\mathrm{E}[A_nA_m^*]\,\mathrm{E}\left[\int_{\mathbb{R}^+}\varphi(T_m+s)\psi^*(T_m)\,dF_S^{(n-m)}(s)\right]\\ =& \sum_{k\geq 1}R_A(k)\sum_{m\geq 1}\mathrm{E}\left[\int_{\mathbb{R}^+}\varphi(T_m+s)\psi^*(T_m)\,dF_S^{(k)}(s)\right]\end{aligned}$$

Remark that Fubini's theorem applies since

$$\begin{aligned}\mathrm{E}\Big[\sum_{n>m\geq 1}|A_nA_m^*|\,|\varphi(T_n)\psi^*(T_m)|\Big] &\leq \mathrm{E}\Big[\sum_{n\geq 1}\sum_{m\geq 1}|A_nA_m^*|\,|\varphi(T_n)\psi^*(T_m)|\Big]\\ &\leq \mathrm{E}\left[|A|^2\right]\mathrm{E}\Big[\sum_{n\geq 1}\sum_{m\geq 1}|\varphi(T_n)\psi^*(T_m)|\Big] = \mathrm{E}\left[|A|^2\right]\mathrm{E}\Big[\sum_{n\geq 1}|\varphi(T_n)|\sum_{m\geq 1}|\psi^*(T_m)|\Big] < \infty\end{aligned}$$

Call $C_A(k) = \operatorname{cov}(A_{n+k}, A_n)$ the covariance of the time series. We can write

$$\begin{aligned}\alpha = \sum_{k\geq 1}\sum_{m\geq 1}C_A(k)\,\mathrm{E}\left[\int_{\mathbb{R}^+}\varphi(T_m+s)\psi^*(T_m)\,dF_S^{(k)}(s)\right]\\ + |\mathrm{E}[A]|^2\sum_{k\geq 1}\sum_{m\geq 1}\mathrm{E}\left[\int_{\mathbb{R}^+}\varphi(T_m+s)\psi^*(T_m)\,dF_S^{(k)}(s)\right].\end{aligned}$$

Since the covariance is summable, applying Fubini's theorem (twice) to the first left term of the above equality yields

$$\begin{aligned}\sum_{m\geq 1}\mathrm{E}\left[\int_{\mathbb{R}^+}\varphi(T_m+s)\psi^*(T_m)\right]\sum_{k\geq 1}C_A(k)\,dF_S^{(k)}(s)&\\ (\text{Fubini}) = \lambda\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\varphi(t+s)\psi^*(t)\,dt\sum_{k\geq 1}C_A(k)\,dF_S^{(k)}(s)&\\ (\text{Parseval}) = \lambda\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}e^{2i\pi\nu s}\widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\,d\nu\sum_{k\geq 1}C_A(k)\,dF_S^{(k)}(s)&\\ (\text{Fubini}) = \lambda\int_{\mathbb{R}^+}\widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\,d\nu\sum_{k\geq 1}C_A(k)\,\phi_S^k(2\pi\nu)&\end{aligned}$$

We then have

$$\operatorname{cov}\left(\sum_{n\in\mathbb{Z}} A_n\varphi(T_n), \sum_{n\in\mathbb{Z}} A_n\psi(T_n)\right) = |\mathrm{E}[A]|^2 \operatorname{cov}\left(\sum_{n\in\mathbb{Z}} \varphi(T_n), \sum_{n\in\mathbb{Z}} \psi(T_n)\right)$$
$$+ \lambda \int_{\mathbb{R}^+} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\, d\nu 2\mathrm{Re}\left\{\sum_{k\geq 1} C_A(k)\phi_S^k(2\pi\nu)\right\} d\nu$$
$$+ \operatorname{Var}(A)\,\mathrm{E}\left[\sum_{n\geq 1}\varphi(T_n)\psi^*(T_n)\right]$$

By definition of Bartlett's spectrum, the first left term of the above equality reads

$$\operatorname{cov}\left(\sum_{n\in\mathbb{Z}} \varphi(T_n), \sum_{n\in\mathbb{Z}} \psi(T_n)\right) = \int_{\mathbb{R}} \widehat{\varphi}(\nu)\widehat{\psi}(\nu)^* \mu_N(d\nu) ,$$

where μ_N is the Bartlett power spectral measure of a renewal process (see for instance [Daley and Vere-Jones, 1988, 2002]), *i.e.*,

$$\mu_N(d\nu) = \lambda\left(2\mathrm{Re}\left\{\sum_{k\geq 0}\phi_S^k(2\pi\nu)\right\} - 1\right) d\nu - \lambda^2\varepsilon_0(d\nu) ,$$

while the third left term reads

$$\mathrm{E}\left[\sum_{n\geq 1}\varphi(T_n)\psi^*(T_n)\right] = \lambda\int_{\mathbb{R}+}\varphi(t)\psi^*(t)\,dt .$$

Assembling all the terms. we obtain

$$\operatorname{cov}\left(\sum_{n\in\mathbb{Z}} A_n\varphi(T_n), \sum_{n\in\mathbb{Z}} A_n\psi(T_n)\right) = |\mathrm{E}[A]|^2 \int_{\mathbb{R}} \widehat{\varphi}(\nu)\widehat{\psi}(\nu)^* \mu_N(d\nu)$$
$$+ 2\lambda \int_{\mathbb{R}^+} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\,\mathrm{Re}\left\{\sum_{k\geq 1} C_A(k)\phi_S^k(2\pi\nu)\right\} d\nu$$
$$+ \operatorname{Var}(A)\,\lambda\int_{\mathbb{R}+}\varphi(t)\psi^*(t)\,dt ,$$

where

$$\sum_{k\geq 1} C_A(k)\phi_S^k(2\pi\nu) = \sum_{k\geq 0} C_A(k)\phi_S^k(2\pi\nu) - \operatorname{Var}(A)$$
$$= \sum_{k\geq 0} R_A(k)\phi_S^k(2\pi\nu) - |\mathrm{E}[A]|^2\sum_{k\geq 0}\phi_S^k(2\pi\nu) - \operatorname{Var}(A) .$$

Then, substituting the expression of the Bartlett spectrum of a renewal process, after rearrangement we obtain result (5.9). □

Remark that condition (5.7) is required for the existence of the spectrum of a renewal process (see for instance [Daley and Vere-Jones, 1988, 2002]). We recall that the latter is given by

$$\mu_N(d\nu) = \lambda\left(2\mathrm{Re}\left\{\sum_{k\geq 0}\phi_S^k(2\pi\nu)\right\} - 1\right)d\nu - \lambda^2\varepsilon_0(d\nu)$$

$$= \begin{cases}\lambda\left(2\mathrm{Re}\left\{\frac{1}{1-\phi_S(2\pi\nu)}\right\} - 1\right)d\nu, & \forall\, \nu \neq 0\,,\\ \lambda\left(1 + \frac{\mathrm{E}[S^2]}{\mathrm{E}[S]}\right)d\nu, & \text{otherwise}\,.\end{cases} \tag{5.10}$$

By a direct comparison of the above formula and (5.9), we can straightfowardly see the spectral contribution of the time series.

5.2 Extended Fundamental Isometry Formula

We now consider the extension of the fundamental isometry formula to the case where the basic point process N is modulated by a process V independent of N. In particular we assume that:

- the basic point process N is simple, of second order and stationary, and that it admits a Bartlett spectrum μ_N with a vector space of test functions B_N;
- that the modulating process V is w.s.s. with Bochner power spectrum μ_V.

Call $\{Y(t)\}_{t\in\mathbb{R}^m}$ a point process N modulated by a stochastic process $\{V(t)\}_{t\in\mathbb{R}^m}$. Recall that (see Section 5.1), its extended Bochner power spectral measure is uniquely defined by

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,V(t)\,N(dt)\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\,\mu_Y(d\nu)\ ,$$

with respect to the domain B_Y. Then, in order to establish a fundamental isometry formula we aim to explicit

$$\mathrm{cov}\left(\int_{\mathbb{R}^m\times K}\varphi(t,z)V(t)\,\bar{N}(dt\times dz)\int_{\mathbb{R}^m\times K}\psi(t,z)V(t)\,\bar{N}(dt\times dz)\right)$$

in terms of the extended Bochner power spectral measure μ_Y. We have the following result.

Theorem 5.2.1 (Extended isometry formula) *Consider the setup of the fundamental isometry formula (Theorem 4.1.1) and of the extended Bochner theorem in the independent case (Theorem 5.1.1). In particular, call B_Y the domain of definition of the extended Bochner spectrum of the point process N modulated by the w.s.s. process $\{V(t)\}_{t\in\mathbb{R}}$. Then, for every $\varphi, \psi : \mathbb{R}^m \times K \to \mathbb{R}$ satisfying conditions (4.1), (4.2) and with expected values ($\bar{\varphi}$ and $\bar{\psi}$) belonging to B_Y, we have*

$$\mathrm{cov}\left(\int_{\mathbb{R}^m\times K}\varphi(t,z)V(t)\,\bar{N}(dt\times dz),\int_{\mathbb{R}^m\times K}\psi(t,z)V(t)\,\bar{N}(dt\times dz)\right)$$
$$= \int_{\mathbb{R}^m}\widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}^*(\nu)\mu_Y(d\nu) + \lambda\mathrm{E}\left[|V|^2\right]\int_{\mathbb{R}^m}\mathrm{cov}\big(\widehat{\varphi}(\nu,Z),\widehat{\psi}^*(\nu,Z)\big)d\nu\,, \tag{5.11}$$

where $\mu_Y = \mu_N * \mu_V + \lambda^2\mu_V + |\mathrm{E}[V]|^2\,\mu_N$.

Proof Since $V(t)$ is a w.s.s. process,

$$\mathrm{E}\left[|V(t)|\right] = \text{const.} < \infty, \quad \text{and} \quad \mathrm{E}\left[|V(t)|^2\right] = \text{const.} < \infty$$

Using Schwarz inequality

$$\mathrm{E}\left[V(t)\, V^*(s)\right] \leq \mathrm{E}\left[|V(t)|^2\right]^{1/2} \mathrm{E}\left[|V(s)|^2\right]^{1/2} = \mathrm{E}\left[|V(t)|^2\right] = \text{const.} < \infty$$

Therefore, since the autocorrelation function is bounden and $V(t)$ and $\bar{N}$ are independent, the exchange of integrals is performed under the same conditions as in the proof of the standard fundamental isometry formula. Hence, we omit the details concerning the application of Fubini's theorem.

We have

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\sum_{(x,z)\in\bar{N}} \varphi(x,z)V(x)\Big)\Big(\sum_{(x,z)\in\bar{N}} \psi^*(x,z)V^*(x)\Big)\Big] \\
&\quad = \mathrm{E}\Big[\sum_{(x,z),(x',z')\in\bar{N},x\neq x'} \varphi(x,z)\psi^*(x',z')V(x)\,V^*\left(x'\right)\Big] + \mathrm{E}\Big[\sum_{(x,z)\in\bar{N}} \varphi(x,z)\psi^*(x,z)\,|V(x)|^2\Big].
\end{aligned}$$

Using nested conditioning with respect to $\mathcal{F}_\infty^N$ and $\mathcal{F}_\infty^V$ we obtain

$$\begin{aligned}
&= \mathrm{E}\Big[\sum_{x,x'\in N,x\neq x'} \bar{\varphi}(x)\bar{\psi}^*(x')V(x)\,V^*\left(x'\right)\Big] + \mathrm{E}\Big[\sum_{x\in N} \mathrm{E}\left[\varphi(x,Z)\psi^*(x,Z)|\mathcal{F}_\infty^N\right] |V(x)|^2\Big] \\
&\quad = \mathrm{E}\Big[\Big(\sum_{x\in N} \bar{\varphi}(x)V(x)\Big)\Big(\sum_{x'\in N} \bar{\psi}^*(x')V^*\left(x'\right)\Big)\Big] - \mathrm{E}\left[\sum_{x\in N} \bar{\varphi}(x)\bar{\psi}^*(x)\,|V(x)|^2\right] \\
&\qquad + \mathrm{E}\Big[\sum_{x\in N} \mathrm{E}\left[\varphi(x,Z)\psi^*(x,Z)|\mathcal{F}_\infty^N\right] |V(x)|^2\Big].
\end{aligned}$$

Since $\mathrm{E}\left[\sum_{(x,z)\in\bar{N}} \varphi(x,z)V(x)\right] = \mathrm{E}\left[\sum_{x\in N} \bar{\varphi}(x)V(x)\right]$,

$$\begin{aligned}
&\mathrm{cov}\Big(\int_{\mathbb{R}^m\times K} \varphi(t,z)V(t)\,\bar{N}(dt\times dz), \int_{\mathbb{R}^m\times K} \psi(t,z)V(t)\,\bar{N}(dt\times dz)\Big) = \\
&\quad \mathrm{cov}\Big(\int_{\mathbb{R}^m} \bar{\varphi}(x)V(x)\,N(dx), \int_{\mathbb{R}^m} \bar{\psi}(x)V(x)\,N(dx)\Big) - \mathrm{E}\Big[\sum_{x\in N} \bar{\varphi}(x)\bar{\psi}^*(x)\,|V(x)|^2\Big] \\
&\qquad + \mathrm{E}\Big[\sum_{x\in N} \mathrm{E}\left[\varphi(x,Z)\psi^*(x,Z)|\mathcal{F}_\infty^N\right] |V(x)|^2\Big].
\end{aligned}$$

Denote by A, B and C the three terms in the right-hand side of the above equation, which then reads $A-B+C$. By definition of the extended Bochner spectrum of a modulated point process (see Section 5.1)

$$A = \int_{\mathbb{R}^m} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}^*(\nu)\mu_{N_V}(d\nu).$$

Using successive conditioning and the independence of $V(t)$ and N

$$B = \mathrm{E}\left[\int_{\mathbb{R}^m} \bar{\varphi}(t)\bar{\psi}^*(t)\,|V(t)|^2\,N(dt)\right] = \mathrm{E}\left[|V(t)|^2\right]\mathrm{E}\left[\int_{\mathbb{R}^m} \bar{\varphi}(t)\bar{\psi}^*(t)N(dt)\right]$$

and by definition of the intensity λ,

$$= \lambda\mathrm{E}\left[|V(t)|^2\right] \int_{\mathbb{R}^m} \bar{\varphi}(t)\bar{\psi}^*(t)dt$$

Similarly

$$C = \lambda \mathrm{E}\left[|V(t)|^2\right] \int_{\mathbb{R}^m} \mathrm{E}\left[\varphi(t,Z)\psi^*(t,Z)\right] dt.$$

By Plancherel-Parseval's identity,

$$\begin{aligned} B =& \lambda \mathrm{E}\left[|V(t)|^2\right] \int_{\mathbb{R}^m} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}(\nu)^* d\nu \\ =& \lambda \mathrm{E}\left[|V(t)|^2\right] \int_{\mathbb{R}^m} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}^*(\nu) d\nu \\ =& \lambda \mathrm{E}\left[|V(t)|^2\right] \int_{\mathbb{R}^m} \mathrm{E}\left[\widehat{\varphi}(\nu,Z)\right] \mathrm{E}\left[\widehat{\psi}^*(\nu,Z)\right] d\nu, \end{aligned}$$

and

$$C = \lambda \mathrm{E}\left[|V(t)|^2\right] \mathrm{E}\left[\int_{\mathbb{R}^m} \widehat{\varphi}(\nu,Z)\widehat{\psi}^*(\nu,Z) d\nu\right],$$

and the result (4.4) follows. □

A straightforward result of the extended fundamental isometry formula is the power spectrum of a modulated shot noise with random excitation (see Definition 2.4).

Corollary 5.2.1 *Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a shot noise with random excitation, where the basic point process N, admitting Bartlett power spectrum μ_N, is modulated by the w.s.s. process $\{V(t)\}_{t\in\mathbb{R}^m}$ with Bochner power spectrum μ_V. Call B_Y the domain of definition of the extended Bochner spectrum of the modulated point process. Assume that the random filtering function h satisfies conditions (4.1), (4.2) and that it has expected value ($\bar{h}$) that belongs to B_Y. Then*

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu,Z)\right]\right|^2 \mu_Y(d\nu) + \lambda \mathrm{E}\left[|V(t)|^2\right] \mathrm{Var}\left(\widehat{h}(\nu,Z)\right) d\nu \tag{5.12}$$

where $\widehat{\ }$ denotes Fourier transformation, and $\mu_Y = \mu_N * \mu_V + \lambda^2\mu_V + |\mathrm{E}[X]|^2 \mu_N$.

Proof Take $\varphi(v,z)$ and $\psi(v,z)$ respectively equal to $h(t-v,z)$ and $h(t+s-v,z)$ in the extended fundamental isometry formula (5.11) to obtain the results. □

The extension of the fundamental isometry formula also applies to the case of a point process N modulated by a time series $\{A_n\}_{n\in\mathbb{Z}}$. We have the following result.

Theorem 5.2.2 (Extended isometry formula - time series) *Consider the setup of the fundamental isometry formula (Theorem 4.1.1) and of the extended Bochner theorem in the case of modulation by time series (Theorem 5.1.3). In particular, call B_Y the domain of definition of the extended Bochner spectrum of the point process N modulated by the w.s.s. time series $\{A_n\}_{n\in\mathbb{Z}}$. Then, for every $\varphi, \psi : \mathbb{R}^m \times K \to \mathbb{R}$ satisfying conditions (4.1), (4.2) and with expected values ($\bar{\varphi}$ and $\bar{\psi}$) belonging to B_Y, we have*

$$\begin{aligned} &\mathrm{cov}\left(\sum_{n\in\mathbb{Z}} A_n\varphi(T_n,Z_n), \sum_{m\in\mathbb{Z}} A_m\psi(T_m,Z_m)\right) \\ &\qquad = \int_{\mathbb{R}^m} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}^*(\nu)\mu_Y(d\nu) + \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \mathrm{cov}\big(\widehat{\varphi}(\nu,Z), \widehat{\psi}^*(\nu,Z)\big) d\nu, \end{aligned} \tag{5.13}$$

where μ_Y is given by (5.9).

Proof First of all remark that, since A_n, $n \in \mathbb{Z}$ is a w.s.s. process,

$$\mathrm{E}\left[|A_n|\right] = \text{const.} < \infty, \quad \text{and} \quad \mathrm{E}\left[|A_n|^2\right] = \text{const.} < \infty$$

Using Schwarz inequality

$$\mathrm{E}\left[A_n A_m^*\right] \leq \mathrm{E}\left[|A|^2\right]^{1/2} \mathrm{E}\left[|A|^2\right]^{1/2} = \mathrm{E}\left[|A|^2\right] = \text{const.} < \infty$$

Therefore, due to the boundedness of the autocorrelation and to the independence of A_n, $n \in \mathbb{Z}$ and $\bar{N}$, the details of the application of Fubini's theorem are the same as in the proof of the standard fundamental isometry formula. Hence, they will be omitted here.

We have

$$\begin{aligned}\mathrm{E}\Big[\sum_{n,m\in\mathbb{Z}} \varphi(T_n, Z_n) A_n \psi^*(T_m, Z_m) A_m^*\Big] =& \mathrm{E}\Big[\sum_{n\neq m\in\mathbb{Z}} \varphi(T_n, Z_n) A_n \psi^*(T_m, Z_m) A_m^*\Big] \\ &+ \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \varphi(T_n, Z_n)\psi(T_n, Z_n)\, |A_n|^2\Big]\end{aligned}$$

using nested conditioning with respect to $\mathcal{F}_\infty^N$ and $\mathcal{F}_\infty^A$ we obtain

$$\begin{aligned}&= \mathrm{E}\Big[\sum_{n\neq m\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) \bar{\psi}^*\left(T_m\right) A_n A_m^*\Big] + \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \mathrm{E}\left[\varphi(T_n, Z)\psi(T_n, Z)|\mathcal{F}_\infty^N\right] |A_n|^2\Big] \\ &= \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) A_n \sum_{m\in\mathbb{Z}} \bar{\psi}^*\left(T_m\right) A_m^*\Big] - \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) \bar{\psi}^*\left(T_n\right) A_n^2\Big] \\ &\quad + \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \mathrm{E}\left[\varphi(T_n, Z)\psi(T_n, Z)|\mathcal{F}_\infty^N\right] |A_n|^2\Big]\end{aligned}$$

Since $\mathrm{E}\left[\sum_{n\in\mathbb{Z}} \varphi(T_n, Z_n)A_n\right] = \mathrm{E}\left[\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) A_n\right]$,

$$\begin{aligned}\mathrm{cov}\Big(\sum_{n\in\mathbb{Z}} A_n\varphi(T_n, Z_n), \sum_{m\in\mathbb{Z}} A_m\psi(T_m, Z_m)\Big) &= \mathrm{cov}\Big(\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) A_n, \sum_{m\in\mathbb{Z}} \bar{\psi}\left(T_m\right) A_m\Big) \\ &- \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) \bar{\psi}^*\left(T_n\right) |A_n|^2\Big] + \mathrm{E}\Big[\sum_{n\in\mathbb{Z}} \mathrm{E}\left[\varphi(T_n, Z)\psi(T_n, Z)|\mathcal{F}_\infty^N\right] |A_n|^2\Big]\end{aligned}$$

Denote by A, B and C the three terms in the right-hand side of the above equation, which then reads $A - B + C$.

By definition of the extended Bochner spectrum of a modulated point process

$$A = \int_{\mathbb{R}^m} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}^*(\nu)\mu_Y(d\nu)\,.$$

Using successive conditioning and the independence of A_n, $n \in \mathbb{Z}$ and N

$$B = \mathrm{E}\left[|A|^2\right] \mathrm{E}\left[\sum_{n\in\mathbb{Z}} \bar{\varphi}\left(T_n\right) \bar{\psi}^*\left(T_n\right)\right]$$

and by definition of the intensity λ,

$$= \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \bar{\varphi}(t)\bar{\psi}^*(t)dt$$

Similarly

$$C = \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \mathrm{E}\left[\varphi(t, Z)\psi^*(t, Z)\right] dt.$$

By Plancherel-Parseval's identity,

$$B = \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \widehat{\bar{\varphi}}(\nu)\widehat{\bar{\psi}}(\nu)^* d\nu = \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \bar{\widehat{\varphi}}(\nu)\bar{\widehat{\psi}}^*(\nu) d\nu$$
$$= \lambda \mathrm{E}\left[|A|^2\right] \int_{\mathbb{R}} \mathrm{E}\left[\widehat{\varphi}(\nu, Z)\right] \mathrm{E}\left[\widehat{\psi}^*(\nu, Z)\right] d\nu,$$

and

$$C = \lambda \mathrm{E}\left[|A|^2\right] \mathrm{E}\left[\int_{\mathbb{R}} \widehat{\varphi}(\nu, Z)\widehat{\psi}^*(\nu, Z) d\nu\right],$$

and the result (5.13) follows. □

Corollary 5.2.2 *Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a shot noise with random excitation, where the basic point process N is a renewal process that admits Bartlett power spectrum μ_N on B_N, and that is modulated by the time series $\{A_n\}_{n\in\mathbb{Z}}$. Consider the set up of Theorem 5.2.2, and in particular, assume that the random filtering function h satisfies conditions (4.1), (4.2) and that it has expected value $(\bar{h})$ that belongs to B_Y. Then*

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu, Z)\right]\right|^2 \mu_Y(d\nu) + \lambda \mathrm{E}\left[|A|^2\right] \mathrm{Var}\left(\widehat{h}(\nu, Z)\right) d\nu \tag{5.14}$$

where $\widehat{\ }$ denotes Fourier transformation, and μ_Y is given by (5.9).

Proof Take $\varphi(v, z)$ and $\psi(v, z)$ respectively equal to $h(t - v, z)$ and $h(t + s - v, z)$ in the extended fundamental isometry formula (5.13) to obtain the results. □

Chapter 6

Branching Point Processes

Summary: This chapter presents the power spectra of the Hawkes branching processes and the generalized birth and death process, the latter being shot noises were the basic point process is a Hawkes branching process and the shots are not independent of the basic process.

Our contribution: We obtain the power spectra of the Hawkes processes in the general spatial case with a non-Poisson "ancestor process", and the power spectra of generalized linear birth and death process. We remark that although the latter are shot noises, their spectrum cannot be obtained from the fundamental isometry formula since the shots now depend on the basic process.

6.1 Hawkes Processes

The Hawkes point process N on $\mathbb{R}^m$ is a spatial *branching point process* [Hawkes, 1971, 1974]. It is defined as

$$N = \sum_{n \geq 0} N_n \tag{6.1}$$

where the sequence of point processes $\{N\}_{n \in \mathbb{N}}$ is constructed as follows.
Let N_0 be a simple second order stationary point process. This point process is called the *ancestors process.*
Now, in order to define $\{N\}_{n \geq 1}$, let $\left\{\bar{N}_n\right\}_{n \geq 0}$ be a sequence of marked point processes with basic point processes $\{N_n\}_{n \geq 0}$ on $\mathbb{R}^m$ and marks $\{Z_n\,(t)\}_{n \geq 0, t \in \mathbb{R}^m}$ with values on the measurable space $(K, \mathcal{K})$ and common distribution Q. Denote by $\mathcal{F}_n$ the sigma-field recording all the events relative to $\bar{N}_0, \ldots, \bar{N}_n$. Then, given a nonnegative *rate function* $h : \mathbb{R}^m \times K \to \mathbb{R}$ such that the quantity

$$\rho := \int_{\mathbb{R}^m} \mathrm{E}\left[h(t, Z)\right] dt$$

is finite (where Z is a K-valued random variable with distribution Q), N_n is, conditionally on $\mathcal{F}_{n-1}$, a Poisson process on $\mathbb{R}^m$ with the intensity

$$\lambda_n(t) = \int_{\mathbb{R}^m \times K} h(t - s, z) \bar{N}_{n-1}(ds \times dz) = \sum_{s \in N_{n-1}} h(t - s, Z_{n-1}(s)). \tag{6.2}$$

N_n is called the *n-th generation point process.*

The interpretation is the following: each point $a \in N_{n-1}$ of generation $n - 1$ creates descendants in the next generation according to a Poisson process of intensity $h(t - a, Z_{n-1}\,(a))$. We therefore have for each ancestor (a point $a \in N_0$) ρ direct descendants on the average.

Define

$$N' = \sum_{n \geq 1} N_n \,.$$

From (6.2) and the Campbell formula we see that, denoting $\lambda_n = \mathrm{E}\left[\lambda_n(t)\right]$,

$$\lambda_n = \lambda_{n-1} \int_{\mathbb{R}^m} \mathrm{E}\left[h(t,Z)\right] dt \,,$$

and therefore the average intensity λ' of N' ($\lambda' = \sum_{n\geq 1} \lambda'_n$) verifies

$$\lambda' = \rho\lambda_0 + \rho\lambda' \,.$$

Therefore, if $\lambda_0 > 0$, in order for N' to have a finite intensity, it is necessary that

$$\rho < 1 \,. \tag{6.3}$$

In this case, each ancestor (point of N_0) is the root of an eventually extinguishing branching process, because its average progeny is strictly less than 1. Condition (6.3) will be assumed throughout.

The Hawkes process corresponds to the diagram in Figure 6.1.

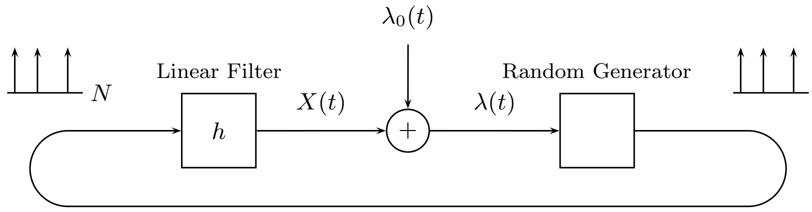


Figure 6.1: Diagram of a Hawkes process.

6.2 Spectra of Hawkes Processes

We now consider the computation of the spectrum of spatial Hawkes point processes. Note that, although Hawkes processes are a particular case of cluster point processes, their spectrum cannot be easily obtained from the general formula (4.17) of the spectrum of cluster point processes (the critical step is the evaluation of $\mathrm{Var}\left(\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,s\rangle} Z\,(ds)\right)$). We follow the approach of Brémaud et al. [2002], where the spectrum was obtained for Hawkes process on $\mathbb{R}$ with Poisson ancestor process.

We shall use the following lemma.

Lemma 6.2.1
A. *Suppose that*

$$\mathrm{E}\left[\left(\int_{\mathbb{R}^m} h\,(t,Z)\,dt\right)^2\right] < \infty. \tag{6.4}$$

There exists, for any given $F \in \mathrm{L}^1_{\mathbb{C}}\left(\ell \times Q\right) \cap \mathrm{L}^2_{\mathbb{C}}\left(\ell \times Q\right)$, *a unique function* $\varphi \in \mathrm{L}^1_{\mathbb{C}}\left(\ell \times Q\right) \cap \mathrm{L}^2_{\mathbb{C}}\left(\ell \times Q\right)$ *such that*

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)\mathrm{E}\left[\varphi(s,Z)\right] ds = F(t,z). \tag{6.5}$$

B. *For given* $f \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$, *there exists a unique* $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$ *such that*

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)\mathrm{E}\left[\varphi(s,Z)\right] ds = f(t)\,. \tag{6.6}$$

Proof

A. For a function $v(t,z)$, denote $\mathrm{E}[v(t,Z)]$ by $\bar{v}(t)$ and $v(-t,z)$ by $\check{v}(t,z)$. Observe that $F \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$ implies $\bar{F} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$. Let h and F be as in the statement of the above lemma, and consider the renewal equation

$$g = \bar{F} + \check{\bar{h}} * g. \tag{6.7}$$

Since $\bar{F} \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$, and since condition (6.3) holds, there exists a unique solution $g \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ given by

$$g = \sum_{n \geq 0} \bar{F} * \check{\bar{h}}^{*n} \tag{6.8}$$

(the convergence of the series in $\mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$ as well as in $\mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ is guaranteed by the inequalities $\|a*b\|_{\mathrm{L}^1} \leq \|a\|_{\mathrm{L}^1}\|b\|_{\mathrm{L}^1}$ and $\|a*b\|_{\mathrm{L}^2} \leq \|a\|_{\mathrm{L}^2}\|b\|_{\mathrm{L}^2}$; uniqueness follows from the equality $g - g' = \check{\bar{h}} * (g-g')$, where g' is another candidate solution, which implies $\|g-g'\|_{\mathrm{L}^1} \leq \|\bar{h}\|_{\mathrm{L}^1}\|g-g'\|_{\mathrm{L}^1}$, and hence under condition (6.3), necessarily $\|g-g'\|_{\mathrm{L}^1} = 0$). The Fourier transform of g is

$$\widehat{g}(\nu) = \frac{\mathrm{E}\left[\widehat{F}(\nu,Z)\right]}{1 - \mathrm{E}\left[\hat{h}(\nu,Z)^*\right]}. \tag{6.9}$$

Define now $\varphi(t,z)$ by

$$\varphi(t,z) = \int_{\mathbb{R}^m} h(s-t,z)\,g(s)\,ds + F(t,z)\,. \tag{6.10}$$

We have

$$\mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|\,dt\right] \leq \mathrm{E}\left[\int_{\mathbb{R}^m} |F(t,Z)|\,dt\right] + \mathrm{E}\left[\int_{\mathbb{R}^m} |h(t,Z)|\,dt\right]\int_{\mathbb{R}^m} |g(t)|\,dt < \infty$$

because $g \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$ and $F, h \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q)$. Therefore $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q)$. We now show that $\varphi \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$. It suffices to show that $\widehat{\varphi}(\nu,z) \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$ because then, using the Plancherel-Parseval identity,

$$\mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|^2\,dt\right] = \mathrm{E}\left[\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu,Z)|^2\,d\nu\right] < \infty.$$

For this purpose, we take for fixed z the Fourier transform of both sides of (6.10)

$$\widehat{\varphi}(\nu,z) = \hat{h}(\nu,z)^*\,\widehat{g}(\nu) + \widehat{F}(\nu,z)\,,$$

or (for future reference) in view of (6.9)

$$\widehat{\varphi}(\nu,z) = \widehat{F}(\nu,z) + \frac{\hat{h}(\nu,z)^*\,\mathrm{E}\left[\widehat{F}(\nu,Z)\right]}{1 - \mathrm{E}\left[\hat{h}(\nu,Z)^*\right]}. \tag{6.11}$$

We show that $\widehat{\varphi}(\nu,z) \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$. Since $F(t,z) \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$, it follows by the Plancherel–Parseval identity that $\widehat{F}(\nu,z) \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$. It remains to show that

$$\hat{h}(\nu,z)\,\hat{g}(\nu) \in \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)\,. \tag{6.12}$$

This follows from the fact that $\widehat{g} \in \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ and that

$$\mathrm{E}\left[\left|\hat{h}(\nu,Z)\right|^2\right] = \mathrm{E}\left[\left|\int_{\mathbb{R}^m} h(t,Z)\,e^{2i\pi\nu t}dt\right|^2\right] \leq \mathrm{E}\left[\left|\int_{\mathbb{R}^m} h(t,Z)\,dt\right|^2\right]$$

a finite constant (by hypothesis (6.4)), independent of ν.

B. This is clearly a particular case of A. We note for future reference that in this case the following holds:

$$\widehat{\varphi}(\nu, z) = \widehat{f}(\nu)\left(1 + \frac{\widehat{h}^*(\nu, z)}{1 - \mathrm{E}\left[\widehat{h}^*(\nu, Z)\right]}\right). \tag{6.13}$$

□

Theorem 6.2.1 (Hawkes processes) *Let $h(t, z)$ verify (6.3) and (6.4). The Bartlett spectrum of N defined in (6.1) is*

$$\mu_N(d\nu) = \frac{1}{\left|1 - \mathrm{E}\left[\hat{h}(\nu, Z)\right]\right|^2}\left[\mu_0(d\nu) + \lambda' d\nu + \lambda \mathrm{Var}\left(\hat{h}(\nu, Z) d\nu\right)\right], \tag{6.14}$$

where $\lambda = \lambda_0/(1-\rho)$. We can take for B_N the set of functions $f \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ such that the solution of (6.6) satisfies $\mathrm{E}[\varphi(\cdot, Z)] \in B_{N_0}$.

Proof We need first the following lemma

Lemma 6.2.2 *For $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$,*

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t, z)\, \bar{M}'(dt \times dz)\right) = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[|\varphi(t, Z)|^2\right] dt, \tag{6.15}$$

where

$$\bar{M}'(dt \times dz) = \bar{N}'(dt \times dz) - \lambda'(t)\, dt\, Q(dz)$$
$$\lambda'(t) = \int_{\mathbb{R}^m}\int_K h(t-s, z')\, \bar{N}_0(dt \times dz') + \int_{\mathbb{R}^m}\int_K h(t-s, z')\, \bar{N}'(dt \times dz')$$

Proof We shall use simplified notation of the kind $\int\int \varphi(t, z)\, \bar{M}'(dt \times dz) = \int \varphi\, d\bar{M}'$. We have

$$\int \varphi\, d\bar{M}' = \sum_{n \geq 1} \int \varphi\, d\bar{M}_n,$$

where $\bar{M}_n(dt \times dz) = \bar{N}_n(dt \times dz) - \lambda_n(t) dt\, Q(dz)$. Given $\mathcal{F}_{n-1}$, $\bar{N}_n$ is a Poisson process with mean measure $\lambda_n(t) Q(dz) dt$, and therefore, by standard properties of Poisson processes,

$$\mathrm{Var}\left(\int \varphi\, d\bar{M}_n \middle| \mathcal{F}_{n-1}\right) = \int_{\mathbb{R}^m} \mathrm{E}\left[\varphi^2(t, Z)\right] \lambda_n(t) dt,$$

and

$$\mathrm{E}\left[\int \varphi\, d\bar{M}_n \middle| \mathcal{F}_{n-1}\right] = 0.$$

Therefore, by the conditional variance formula

$$\begin{aligned}\mathrm{Var}\left(\int \varphi\, d\bar{M}_n\right) &= \mathrm{E}\left[\mathrm{Var}\left(\int \varphi\, d\bar{M}_n \middle| \mathcal{F}_{n-1}\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\int \varphi\, d\bar{M}_n \middle| \mathcal{F}_{n-1}\right]\right) \\ &= \lambda_n \int_{\mathbb{R}^m} \mathrm{E}\left[\varphi^2(t, Z)\right] dt.\end{aligned}$$

Also for $j, k \geq 1$,

$$\mathrm{E}\left[\left(\int \varphi\, d\bar{M}_j\right)\left(\int \varphi\, d\bar{M}_{j+k}\right)\right] = \mathrm{E}\left[\left(\int \varphi\, d\bar{M}_j\right) \mathrm{E}\left[\int \varphi\, d\bar{M}_{j+k} \middle| \mathcal{F}_{j+k-1}\right]\right] = 0.$$

Therefore

$$\begin{aligned}\mathrm{Var}\left(\int \varphi\, d\bar{M}'\right) &= \sum_{n\geq 1}\mathrm{Var}\left(\int \varphi\, d\bar{M}_n\right)\\ &= \left(\sum_{n\geq 1}\lambda_n\right)\int_{\mathbb{R}^m}\mathrm{E}\left[\varphi^2(t,Z)\right] dt\\ &= \lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[\varphi^2(t,Z)\right] dt.\end{aligned}$$

□

Let φ be the solution of (6.6).

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz) &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\left(\bar{N}'(dt\times dz)-\lambda'(t)Q(dz)dt\right)\\ &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}'(dt\times dz) - \int_{\mathbb{R}^m}\int_K \varphi(t,z)\lambda'(t)Q(dz)dt.\end{aligned}$$

Also,

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\lambda'(t)Q(dz)dt &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\left(\int_{\mathbb{R}^m}\int_K h(t-s,z')\bar{N}(ds\times dz')\right)Q(dz)dt\\ &= \int_{\mathbb{R}^m\times\mathbb{R}^m\times K} h(t-s,z')\mathrm{E}\left[\varphi(t,Z)\right] dt\bar{N}(ds\times dz')\\ &= \int_{\mathbb{R}^m}\int_K \left(\check{h}(s-\cdot,z')*\mathrm{E}\left[\varphi(\cdot,Z)\right]\right)(s)\,\bar{N}(ds\times dz').\end{aligned}$$

Therefore, since $\bar{N}=\bar{N}'+\bar{N}_0$,

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz) &= \int_{\mathbb{R}^m}\int_K \left(\varphi(t,z)-\left(\check{h}(t-\cdot,z)*\mathrm{E}\left[\varphi(\cdot,Z)\right]\right)(t)\right)\bar{N}(dt\times dz)\\ &\quad -\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz).\end{aligned}$$

Take $f\in B_N$ (in particular $f\in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)\cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$), and let $\varphi(t,z)$ be the solution of (6.6). We have

$$\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)+\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)=\int_{\mathbb{R}^m} f(t)N(dt).$$

Also, by the isometry lemma,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)\right)=\lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[|\varphi(t,Z)|^2\right] dt=\lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[|\widehat{\varphi}(\nu,Z)|^2\right] d\nu.$$

Now,

$$\begin{aligned}&\mathrm{E}\left[\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right]\\ &\qquad = \mathrm{E}\left[\mathrm{E}\left[\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)\middle|\mathcal{F}_0\right]\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right]=0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)+\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right)=\\ &\qquad \mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{M}'(dt\times dz)\right)+\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right)=\\ &\qquad\qquad \lambda'\int_{\mathbb{R}^m}\mathrm{E}\left[|\varphi(t,Z)|^2\right] dt+\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right).\end{aligned}$$

On the other hand,

$$\text{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\bar{N}_0(dt\times dz)\right) = \int_{\mathbb{R}^m} |\text{E}\left[\widehat{\varphi}(\nu,Z)\right]|^2 \mu_0(d\nu) + \lambda_0 \int_{\mathbb{R}^m} \text{Var}\left(\widehat{\varphi}(\nu,Z)\right) d\nu.$$

Combining the above, we have

$$\begin{aligned}\text{Var}\left(\int_{\mathbb{R}^m} f(t)N(dt)\right) &= \lambda' \int_{\mathbb{R}^m} \text{E}\left[|\widehat{\varphi}(\nu,Z)|^2\right] d\nu + \int_{\mathbb{R}^m} |\text{E}\left[\widehat{\varphi}(\nu,Z)\right]|^2 \mu_0(d\nu) \\ &\quad + \lambda_0 \int_{\mathbb{R}^m} \text{Var}\left(\widehat{\varphi}(\nu,Z)\right) d\nu \\ &= A + B + C.\end{aligned}$$

By Formula (6.13),

$$\begin{aligned} A &= \lambda' \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \frac{1+\text{Var}\left(\hat{h}(\nu,Z)\right)}{|1-\text{E}\left[\hat{h}(\nu,Z)\right]|^2} d\nu, \\ B &= \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \frac{1}{|1-\text{E}\left[\hat{h}(\nu,Z)\right]|^2} \mu_0(d\nu), \\ C &= \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \lambda_0 \frac{\text{Var}\left(\hat{h}(\nu,Z)\right)}{|1-\text{E}\left[\hat{h}(\nu,Z)\right]|^2} d\nu. \end{aligned}$$

Recalling that $\lambda' = \rho\lambda_0/(1-\rho)$, we obtain finally that for all $f \in \text{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \text{L}^2_{\mathbb{C}}(\mathbb{R}^m)$,

$$\text{Var}\left(\int_{\mathbb{R}^m} f(t)N(dt)\right) = \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \left(\frac{1}{|1-\text{E}\left[\hat{h}(\nu,Z)\right]|^2}\right)\left(\mu_0(d\nu) + \lambda' d\nu + \lambda \text{Var}\left(\hat{h}(\nu,Z)\right) d\nu\right),$$

and this allows us to identify μ_N as (6.14). □

EXAMPLE 6.1: The original Hawkes process. In the particular case

$$h(t,Z) = h(t),$$

$\hat{h}(\nu,Z) = \hat{h}(\nu)$, and we have

$$\mu_N(d\nu) = \frac{1}{|1-\hat{h}(\nu)|^2}\left[\mu_0(d\nu) + \lambda' d\nu\right].$$

If in addition N_0 is a Poisson process with average intensity α, since $\alpha + \lambda' = \lambda$, we have the original formula of Hawkes

$$\mu_N(d\nu) = \frac{\lambda d\nu}{|1-\hat{h}(\nu)|^2}.$$

6.3 Birth and Death Processes as Shot Noise

The generalized linear birth and death process (not necessarily Markovian), are shot noises where the basic point process is a Hawkes process and the sequence of pulses is *not* independent of the basic point process. They are described as

$$X(t) = \sum_{s \in N} \alpha(t - s, Z(s)). \tag{6.16}$$

Note that its spectral characteristics cannot be derived from Theorem 4.1.1, since now the marks $Z(s)$ and the process N are not independent.

Example 6.2: Univariate case. In the univariate case $\mathbb{R}^m = \mathbb{R}$,

$$X(t) = \sum_{n \in \mathbb{Z}} \alpha(t - T_n, Z_n).$$

To further specialize this example, take

$$h(t, z) = \beta 1_{[0,z]}(t),$$

and

$$\alpha(t, z) = 1_{[0,z]}(t).$$

Therefore interpreting T_n as the birth time of individual n in the colony, and Z_n as its lifetime,

$$X(t) = \sum_{n \in \mathbb{Z}} 1_{(-\infty,t]}(T_n) 1_{(t,+\infty)}(T_n + Z_n)$$

is the number of individuals in the colony.

If moreover we assume that Z_n is exponentially distributed with parameter γ, and that the process N_0 of ancestors is Poisson with intensity λ_0, the process $\{X(t)\}_{t \in \mathbb{R}}$ is a Markov birth and death process with infinitesimal generator Q given by its non-null terms $q_{i,i+1} = \lambda_0 + \beta i$, $q_{i,i-1} = \gamma i$.

Theorem 6.3.1 (Birth and Death Processes) *Consider the process $\{X(t)\}$ defined by (6.16), where $\alpha \in L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$, and where the conditions stated in Theorem 6.2.1 are satisfied for N. Suppose moreover that the solution φ of (6.5) of Lemma 6.2.1 for $F(s,z) = \int_{\mathbb{R}^m} \alpha(t-s,z) f(t)\, dt$ is such that $E[\varphi(\cdot, Z)] \in B_{N_0}$ for any $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)$. Then the Bochner spectral measure μ_X is given by the expression*

$$\begin{aligned} \left|1 - \mathrm{E}\left[\hat{h}(\nu, Z)\right]\right|^2 \mu_X(d\nu) &= |\mathrm{E}[\widehat{\alpha}(\nu, Z)]|^2 \left(1 + \frac{\rho\lambda_0}{1-\rho}\right) \mu_0(d\nu) \\ &+ \frac{\lambda_0}{1-\rho} \mathrm{Var}\left\{\widehat{\alpha}(\nu, Z)\left(1 - \mathrm{E}\left[\hat{h}(\nu, Z)\right]\right) + \hat{h}(\nu, Z)\, \mathrm{E}[\widehat{\alpha}(\nu, Z)]\right\} d\nu. \end{aligned} \tag{6.17}$$

Proof We seek a measure μ_X such that for all $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)$

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} f(t) X(t)\, dt\right) = \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \mu_X(d\nu). \tag{6.18}$$

But

$$\int_{\mathbb{R}^m} f(t) X(t)\, dt = \int_{\mathbb{R}^m} f(t) \left(\int_{\mathbb{R}^m} \int_K \alpha(t-s,z) \bar{N}(ds \times dz) \right) dt$$
$$= \int_{\mathbb{R}^m} \int_K F(s,z) \bar{N}(ds \times dz),$$

where

$$F(s,z) = \int_{\mathbb{R}^m} \alpha(t-s,z) f(t)\, dt$$
$$= (\breve{\alpha}(\cdot,z) * f)(s)$$

is a function in $\mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$ (because $\alpha \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$, and $f \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m)$). Therefore we seek μ_X such that

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \int_K F(s,z) \bar{N}(ds \times dz) \right) = \int_{\mathbb{R}^m} \left| \widehat{f}(\nu) \right|^2 \mu_X(d\nu).$$

Following the same calculations as in the proof of Theorem 6.2.1 up to the 3rd displayed equation thereof, and letting φ be the unique solution in $\mathrm{L}^1_{\mathbb{C}}(\ell \times Q) \cap \mathrm{L}^2_{\mathbb{C}}(\ell \times Q)$ of equation (6.5) of Lemma 6.2.1, we have

$$\int_{\mathbb{R}^m} \int_K F(s,z) \bar{N}(ds \times dz) = \int_{\mathbb{R}^m} \int_K \varphi(t,z) \bar{M}'(dt \times dz) + \int_{\mathbb{R}^m} \int_K \varphi(t,z) \bar{N}_0(dt \times dz).$$

Resuming the proof of Theorem 6.2.1 after the 4th displayed equation thereof, we obtain

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} f(t) X(t)\, dt \right) = \lambda' \int_{\mathbb{R}^m} \mathrm{E}\left[|\widehat{\varphi}(\nu,Z)|^2 \right] d\nu + \int_{\mathbb{R}^m} |\mathrm{E}[\widehat{\varphi}(\nu,Z)]|^2 \mu_0(d\nu)$$
$$+ \lambda_0 \int_{\mathbb{R}^m} \mathrm{Var}(\widehat{\varphi}(\nu,Z))\, d\nu$$
$$= A + B + C$$

where, using the expression for $\widehat{\varphi}(\nu,z)$

$$\widehat{\varphi}(\nu,z) = \widehat{F}(\nu,z) + \frac{\hat{h}(\nu,z)^* \mathrm{E}\left[\widehat{F}(\nu,Z) \right]}{1 - \mathrm{E}\left[\hat{h}(\nu,Z)^* \right]}$$
$$= \widehat{f}(\nu) \left[\widehat{\alpha}(\nu,z)^* + \frac{\hat{h}(\nu,z)^* \mathrm{E}[\widehat{\alpha}(\nu,Z)^*]}{1 - \mathrm{E}\left[\hat{h}(\nu,Z)^* \right]} \right]$$

we find that

$$A = \lambda' \int_{\mathbb{R}^m} \left| \widehat{f}(\nu) \right|^2 \frac{\mathrm{E}\left[\left| \widehat{\alpha}(\nu,Z) \left(1 - \mathrm{E}\left[\hat{h}(\nu,Z) \right] \right) + \hat{h}(\nu,Z) \mathrm{E}[\widehat{\alpha}(\nu,Z)] \right|^2 \right]}{\left| 1 - \mathrm{E}\left[\hat{h}(\nu,Z) \right] \right|^2} d\nu,$$

$$B = \int_{\mathbb{R}^m} \left| \widehat{f}(\nu) \right|^2 \frac{|\mathrm{E}[\widehat{\alpha}(\nu,Z)]|^2}{\left| 1 - \mathrm{E}\left[\hat{h}(\nu,Z) \right] \right|^2} \mu_0(d\nu),$$

$$C = \lambda_0 \int_{\mathbb{R}^m} \left| \widehat{f}(\nu) \right|^2 \frac{\mathrm{Var}\left(\widehat{\alpha}(\nu,Z) \left(1 - \mathrm{E}\left[\hat{h}(\nu,Z) \right] \right) + \hat{h}(\nu,Z) \mathrm{E}[\widehat{\alpha}(\nu,Z)] \right)}{\left| 1 - \mathrm{E}\left[\hat{h}(\nu,Z) \right] \right|^2} d\nu.$$

Therefore, using the expression $\lambda' = \rho\lambda_0/(1-\rho)$, we find after rearrangement (6.18) with $\mu_X(d\nu)$ given by (6.17). $\square$

EXAMPLE 6.3: Consider the set-up of the previous example, i.e. that $\{X(t)\}_{t\in\mathbb{R}}$ is a Markov birth and death process with non-null transition rates $q_{i,i+1} = \lambda_0 + \beta i$, $q_{i,i-1} = \gamma i$. We then have the following identifications:

$$\rho = \beta/\gamma\,, \quad \mu_0(d\nu) = \lambda_0 d\nu\,,$$
$$\widehat{\alpha}(\nu,z) = \frac{1-e^{-2i\pi\nu z}}{2i\pi\nu}, \quad \widehat{h}(\nu,z) = \beta\frac{1-e^{-2i\pi\nu z}}{2i\pi\nu}.$$

From this we obtain

$$\mathrm{E}\left[\widehat{h}(\nu,Z)\right] = \beta\mathrm{E}\left[\widehat{\alpha}(\nu,Z)\right]\,, \quad \mathrm{E}\left[\widehat{\alpha}(\nu,Z)\right] = \frac{1}{2\pi i\nu}\left(1-\frac{1}{\gamma+2\pi i\nu}\right)\,,$$

$$\left|\mathrm{E}\left[\widehat{\alpha}(\nu,Z)\right]\right|^2 = \frac{1}{4\pi^2\nu^2\left(\gamma^2+4\pi^2\nu^2\right)^2}\left(\left(\gamma^2+4\pi^2\nu^2-\gamma\right)^2+4\pi^2\nu^2\right)\,,$$
$$\left|1-\mathrm{E}\left[\widehat{h}(\nu,Z)\right]\right|^2 = \frac{1}{4\pi^2\nu^2\left(\gamma^2+4\pi^2\nu^2\right)^2}\left(4\pi^2\nu^2\left(\gamma^2+4\pi^2\nu^2-\beta\right)^2+\beta^2\left(\gamma^2+4\pi^2\nu^2-\gamma\right)^2\right)\,,$$

and

$$\widehat{\alpha}\left(\nu,Z\right)\left(1-\mathrm{E}\left[\hat{h}\left(\nu,Z\right)\right]\right)+\hat{h}\left(\nu,Z\right)\mathrm{E}\left[\widehat{\alpha}\left(\nu,Z\right)\right] = \widehat{\alpha}(\nu,Z)\,,$$
$$\mathrm{Var}\left(\widehat{\alpha}\left(\nu,Z\right)\right) = \frac{1}{4\pi^2\nu^2}\left(1-\frac{1}{\gamma^2+4\pi^2\nu^2}\right)\,.$$

Combined with Formula (6.17), this then yields the following spectral measure:

$$\mu_X(d\nu) = \frac{1}{\left(4\pi^2\nu^2\left(\gamma^2+4\pi^2\nu^2-\beta\right)^2+\beta^2\left(\gamma^2+4\pi^2\nu^2-\gamma\right)^2\right)}\times$$
$$\left\{\left(\left(\gamma^2+4\pi^2\nu^2-\gamma\right)^2+4\pi^2\nu^2\right)\left(\lambda_0+\frac{\beta}{\gamma-\beta}\lambda_0^2\right)+\right.$$
$$\left.\frac{\lambda_0\gamma}{\gamma-\beta}\left(\gamma^2+4\pi^2\nu^2\right)\left(\gamma^2+4\pi^2\nu^2-1\right)\right\}$$

Chapter 7

UWB Signals

Summary: The large family of UWB signals is aptly modelled as shot noises: each specific signal is constructed by adding features to a basic model. Their power spectrum is then computed using the formulas we have previously derived for complex signals related to filtered spike fields.

Our contribution: We propose a unifying shot noise model that is modular, simple and tractable. The spectrum can be computed form a singular general formula, providing simpler proofs for existing results and general spectrum expressions where the various features of the model appear explicitly. These features have a tremendous impact on the design and the analisys of UWB signals.

The family of pulse modulation techniques has gathered increasing attention since the introduction of Ultrawide Bandwidth (UWB) impulse radio [Scholtz, 1993; Win and Scholtz, 1998; Siwiak, 2001].

The characteristic of UWB radio is to communicate with pulses of very short duration, thereby spreading the energy of the radio signal over several GHz. Signals are transmitted with extremely low spectral content while maintaining the average power level required for reliable communications. As a consequence, the spectral characteristics (spectral occupancy and composition) of an UWB transmission has a key role in the design of UWB systems.

Here we show that the large family of UWB signals are aptly modelled as shot noises with random excitation (or related complex signal), where:

- the underlying point process determines the temporal structure of the signal;
- the random function characterizes the shape of the pulses.

There are two considerable advantages in modeling UWB signals as shot noises. Firstly, the model is *modular*, *simple* and *tractable*, and it allows to:

- construct different UWB signals in a unifying way by adding features to a basic model;
- easily take into account random quantities affecting the signals, such as jitter, losses or pulse distortions.

Secondly, spectra are obtained, in a systematic and rigorous manner, from a singular general formula that provides:

- simpler proofs for existing results;
- spectrum expression of highly complexified signals where the various features of the model appears explicitly.

These advantages have a tremendous impact to the design and the analisys of UWB signal models.

Here we focus on the following UWB signals

- pulse amplitude modulation - PAM, and pulse position modulation - PPM;

- (digital) pulse interval modulation - DPIM, and (digital) pulse interval and amplitude modulation - DPIAM;
- time-hopping signals - TH , and direct sequence signals - DS.

Pulse modulations provide a support for the information, while time-hopping and direct sequence signals allows multiple access transmissions. Commonly, a UWB transmission employs either time-hopping or direct sequence techniques to achieve multiple access, and either pulse position or pulse amplitude modulation for data transmission (see for instance [Win and Scholtz, 1998, 2000]).

We consider one-dimensional models of signals taking real values. Extension to signals with complex values is straightforward while the extension to multi dimensions can be easily achieved following the general definitions of point process and related process on $\mathbb{R}^m$ given in Chapter 1 and Chapter 2. In mathematical terms, UWB signals are therefore random processes of the form (2.2), *i.e.*,

$$X(t) = \int_{\mathbb{R}\times K} h(t-s,z)\bar{N}(ds \times dz)\,, \tag{7.1}$$

or, in the more general case where the basic point process of the shot noise is modulated by a time series $\{A_n\}_{n\in\mathbb{Z}}$, of the form (2.7), *i.e.*,

$$X(t) = \sum_{n\in\mathbb{Z}} A_n h\left(t - T_n, Z_n\right)\,, \quad t \in \mathbb{R}\,. \tag{7.2}$$

We remark that the former is a special case of the latter with $A_n \equiv 1$, $n \in \mathbb{Z}$. We recall that the spectrum of (7.1) can be computed using the fundamental isometry formula (4.4), and in particular using equation (4.8), *i.e.*,

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu,Z)\right]\right|^2 \mu_N(d\nu) + \lambda \mathrm{Var}\left(\widehat{h}(\nu,Z)\right) d\nu\,. \tag{7.3}$$

while, in the case of a shot noise with basic renewal point process modulated by a time series (7.2), the spectrum is given by (5.14), *i.e.*,

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu,Z)\right]\right|^2 \mu_Y(d\nu) + \lambda \mathrm{E}\left[|A|^2\right] \mathrm{Var}\left(\widehat{h}(\nu,Z)\right) d\nu \tag{7.4}$$

where μ_Y is given by (5.9). Here, we can avoid to deal with the technicalities related to the domain B_N of definition of the Bartlett spectrum since the filtering functions, *i.e.* the pulse shapes, that appears in UWB signals belongs to the class regular bounded functions that rapidly decrease or have finite support, and such a class in contained in B_N (see Chapter 3).

7.1 Pulse Position Modulation

In pulse position modulation (PPM) the information is carried by the relative distance of the pulses with respect to a regular grid (see for instance [Middleton, 1960; Schwartz et al., 1966; Win and Scholtz, 1998, 2000; Siwiak, 2001]). Therefore, a PPM signal can be seen as a regularly spaced point process N convoluted with the pulse shape, to which we add positive random i.i.d. displacement coding the symbols to be transmitted. The points of N are given by

$$T_n = U + nT, \quad n \in \mathbb{N}\,,$$

where U is a uniform-$[0,T]$ random variable, that ensures the stationarity of N, and T is the period of repetition of the spikes. The random displacements, coding the symbols, are modelled using the i.i.d. marks $\{Z_n\}_{n\in\mathbb{Z}}$. Calling $w(t)$, $t \in \mathbb{R}$, the pulse shape, a PPM signal is modelled as

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - U - nT - Z_n) \ , \tag{7.5}$$

Hence, we have the shot noise with random excitation of equation (7.2), where

- $A_n := 1$, $n \in \mathbb{Z}$;
- $T_n := U + nT$, $n \in \mathbb{Z}$, with U $[0, T]$-uniformly distributed.
- $h(t, Z) := w(t - Z)$

Figure 7.1 shows an example of pulse position modulation.

EXAMPLE 7.1: PPM with jitter. Following Example 2.4 (shot noise with jitter) and equation (7.5), a pulse position modulation in the presence of jitter is a shot noise with random excitation with filtering function

$$h(t, Z) := w(t - Z^{\text{P}} - Z^{\text{J}}) \ ,$$

i.e., the sequence of i.i.d. marks is now the random vector $Z_n := (Z_n^{\text{P}}, Z_n^{\text{J}})$, $n \in \mathbb{Z}$, where Z^{P} models the transmitted symbols and Z^{J} the jitter.

Power Spectrum

The basic point process of the shot noise modelling a PPM signal has power spectral pseudo density (see Example 3.1)

$$S_N(\nu) = \frac{1}{T^2} \sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) \ .$$

Notice that now the average number of points λ is $1/T$. Since the pulse displacements are modeled in the same way as the jitter, we can apply the result of Corollary 4.2.4 to obtain the power spectral pseudo density of a PPM signal

$$S_X(\nu) = |\widehat{w}(\nu)|^2 |\phi_Z(2\pi\nu)|^2 \frac{1}{T^2} \sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) + \frac{1}{T} |\widehat{w}(\nu)|^2 \left(1 - |\phi_Z(2\pi\nu)|^2\right) \ ,$$

where now ϕ_Z is the characteristic function of the transmitted symbols.

The effect of jittering and thinning can be straightforwardly taken into account. For instance, if the transmission is affected by i.i.d. losses, we consider a vector of marks $Z = (Z^{\text{P}}, Z^{\text{L}})$ where now Z^{P} and Z^{L} model, respectively, the random displacements (coding the information) and the random losses. Then,

$$S_X(\nu) = |\widehat{w}(\nu)|^2 \,\mathrm{E}\left[{Z^{\text{L}}}^2\right] |\phi_Z(2\pi\nu)|^2 \frac{1}{T^2} \sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) + \frac{1}{T} |\widehat{w}(\nu)|^2 \left(\mathrm{E}\left[{Z^{\text{L}}}^2\right] - \mathrm{E}\left[Z^{\text{L}}\right]^2 |\phi_Z(2\pi\nu)|^2\right) \ .$$

EXAMPLE 7.2: PPM with M i.i.d. symbols. For instance, if we consider the transmission of M independent symbols, each with equal probability $1/M$, the spikes take, with equal probability, the relative positions $\{0, T/M, \ldots, T(M-1)/M\}$. Therefore, the characteristic function of the "jitters" is given by

$$\phi_Z(2\pi\nu) = \frac{1}{M} e^{i\pi\nu T \frac{M-1}{M}} \frac{\sin(\pi\nu T)}{\sin(\pi\nu T/M)} .$$

The spectrum of a PPM signal is a known result. However, using our modular approach, the model can be complexified *ad libitum* obtaining several extensions of known results.

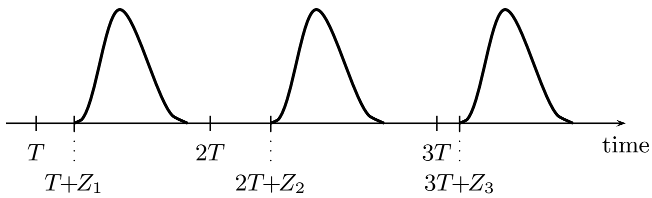


Figure 7.1: Pulse position modulation (PPM), where the marks model the coded symbols.

7.2 Pulse Amplitude Modulation

A pulse amplitude modulation (PAM) transmits the information through the amplitude of the pulses (see for instance [Middleton, 1960; Schwartz et al., 1966]). Hence we have a regularly spaced basic point process N convoluted with a pulse shape with random amplitude. When the symbols are coded into a sequence of i.i.d. amplitudes, a PAM signal can be modelled as

$$X(t) = \sum_{n\in\mathbb{Z}} Z_n w(t - U - nT) , \tag{7.6}$$

i.e., the shot noise of equation (7.2) with

- $A_n := 1$, $n \in \mathbb{Z}$;
- $T_n := U + nT$, $n \in \mathbb{Z}$.
- $h(t, Z) := Zw(t)$

as depicted in Figure 7.2. Otherwise, if we consider correlated amplitudes we have

$$X(t) = \sum_{n\in\mathbb{Z}} A_n w(t - U - nT) , \tag{7.7}$$

i.e., the shot noise of equation (7.2) with

- $T_n := U + nT$, $n \in \mathbb{Z}$.
- $h(t, Z) := w(t)$;

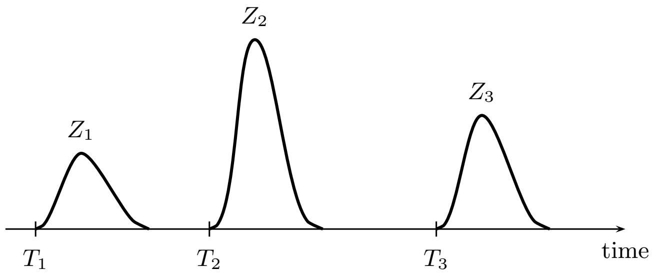


Figure 7.2: Pulse amplitude modulation with i.i.d. amplitudes.

EXAMPLE 7.3: PAM with random losses. We consider a PAM signal with i.i.d. amplitudes (the extension to the case of correlated amplitudes is straightforward). Following Example 2.2 (shot noise with random losses) and equation (7.6), a pulse amplitude modulation in the presence of random losses of the pulses is a shot noise with random excitation with filtering function

$$h(t, Z) := Z^{\mathrm{A}} Z^{\mathrm{L}} w(t) ,$$

i.e., the sequence of i.i.d. marks is now the random vector $Z_n = (Z_n^{\mathrm{A}}, Z_n^{\mathrm{L}})$, $n \in \mathbb{Z}$, where Z^{A} models the transmitted symbols while Z^{L}, with values in $\{0, 1\}$, models the random losses.

Power Spectrum

For i.i.d. amplitudes, the PAM model is given by equation (7.6). Hence the basic point process is a regular grid with $\lambda = 1/T$ and power spectral pseudo density

$$S_N(\nu) = \frac{1}{T^2} \sum_{n \neq 0} \delta\left(\nu - \frac{n}{T}\right) .$$

The filtering function is $h(t, Z) := Zw(t)$, with

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\, \mathrm{E}[Z] , \quad \mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = |\widehat{w}(\nu)|^2\, \mathrm{E}\left[|Z|^2\right] .$$

The power spectral pseudo then reads

$$S_X(\nu) = |\widehat{w}(\nu)|^2\, |\mathrm{E}[Z]|^2 \frac{1}{T^2} \sum_{n \neq 0} \delta\left(\nu - \frac{n}{T}\right) + \frac{1}{T} |\widehat{w}(\nu)|^2\, \mathrm{Var}(Z) .$$

When the amplitudes are correlated the PAM model is given by equation (7.7). We then have a shot noise where the basic point process has regularly spaced points modulated by the time series. By remarking that a regular grid is a particular case of a renewal process, we apply formula (7.4) to obtain

$$S_X(\nu) = |\widehat{w}(\nu)|^2 \left(\frac{1}{T} S_A(T\nu) + \frac{1}{T^2} |\mathrm{E}[A]|^2 \sum_{n \in \mathbb{Z}} \delta\left(\nu - \frac{n}{T}\right)\right) ,$$

where here S_A is the spectral density of the time series. The above formula is also known as the "Bennett-Rice" formula [Bennett and Rice, 1963].

Here again, using our modular approach, the model can be complexified *ad libitum* obtaining various original extensions of known results. For instance, using formula (4.13) of a shot noise with jitter, we straightforwardly obtain the spectrum of a PAM signal with correlated amplitude and i.i.d. jitter

$$S_X(\nu) = |\phi_Z(\nu)|^2 |\widehat{w}(\nu)|^2 \left(\frac{1}{T} S_A(T\nu) + \frac{1}{T^2} |\mathrm{E}[A]|^2 \sum_{n\in\mathbb{Z}} \delta\left(\nu - \frac{n}{T}\right)\right) + \frac{1}{T} |\widehat{w}(\nu)|^2 \left(1 - |\phi_Z(\nu)|^2\right) d\nu \,.$$

7.3 Pulse Interval Modulations

In pulse interval modulated signals (DPIM) the information is coded with the relative distance between successive pulses [Ghassemlooy et al., 1996, 1998]. The sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$ is then a renewal point process (see example 1.1). The DPIM is then the convolution of such a point process with the pulse shape

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - T_n) \,.$$

Thus, a shot noise (7.2) with

- $A_n := 1$, $n \in \mathbb{Z}$;
- $\{T_n\}_{n\in\mathbb{Z}}$, is a renewal process.
- $h(t, Z) := w(t)$;

(pulse interval and amplitude modulated signals - DPIAM, are discussed in the following section).

Power Spectrum

The power spectrum of a DPIM signal is a simple results since it is given by the power spectrum of a renewal process (a well know result; see for instance (5.10)), times the Fourier transform of the pulse shape, *i.e.*,

$$S_X(\nu) = |\widehat{w}(\nu)|^2 \lambda \left(2\mathrm{Re}\left\{ \sum_{k\geq 0} \phi_S^k(2\pi\nu) \right\} d\nu - 1 - \lambda\delta(\nu) \right) \,.$$

where ϕ_S is the characteristic function of the inter-arrival times (see Example 1.1).

EXAMPLE 7.4: Dpim for the transmission of i.i.d. symbols. Consider the transmission of i.i.d. symbols from an alphabet with M values, uniformly distributed. Such symbols can be coded with the relative distance between two pulses. Hence, is is a shot noise where the basic point process is a renewal process with i.i.d. inter-arrivals $\{S_n\}_{n\in\mathbb{Z}}$ uniformly taking values over $\{T, 2T, \ldots, MT\}$, where T is the basic increment of the relative distance. Then, the spectrum is given by the above formula with

$$\phi_S(2\pi\nu) = \frac{1}{M} e^{i\pi\nu TM} \frac{\sin(\pi\nu T(M+1))}{\sin(\pi\nu T)} \,.$$

7.4 Combination of Pulse Modulations

Any combination of the modulations we have exposed can be easily taken into account. In particular, the spectrum of combined modulations is straightforwardly computed from the spectrum of the basic modulations. This remarkable feature is a direct consequence of the modular approach and it is one of the most important aspects of our contribution.

We shall present some common situations.

EXAMPLE 7.5: Pulse position and amplitude modulation - PPM/PAM. With i.i.d. amplitudes

$$X(t) = \sum_{n\in\mathbb{Z}} Z_n^{\mathrm{A}} w\left(t - U - nT - Z_n^{\mathrm{P}}\right),$$

i.e., equation (2.7) with

- $A_n := 1$, $n \in \mathbb{Z}$;
- $T_n = U + nT$, $n \in \mathbb{Z}$;
- $Z := (Z^{\mathrm{A}}, Z^{\mathrm{P}})$;
- $h(t, Z) := Z^{\mathrm{A}} w\left(t - Z^{\mathrm{P}}\right)$.

Power Spectrum

We have

$$\widehat{h}(\nu, Z) = \widehat{w}(\nu)\, Z^{\mathrm{A}} e^{-i2\pi\nu Z^{\mathrm{P}}},$$

and

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\,\mathrm{E}\left[Z^{\mathrm{A}}\right]\phi_{Z^{\mathrm{P}}}(-2\pi\nu), \quad \mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = \left|\widehat{w}(\nu)\right|^2 \mathrm{E}\left[\left|Z^{\mathrm{A}}\right|^2\right],$$

where $\phi_{Z^{\mathrm{P}}}$ is the characteristic function of the i.i.d. displacements coding the symbols to be transmitted. The basic point process is a regular grid with $\lambda = 1/T$ and spectrum

$$S_N(\nu) = \frac{1}{T^2}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right).$$

Hence, the power spectrum of a PPM/PAM signal is given by

$$S_X(\nu) = \left|\widehat{w}(\nu)\right|^2 \left|\mathrm{E}\left[Z^{\mathrm{A}}\right]\right|^2 \left|\phi_{Z^{\mathrm{P}}}(2\pi\nu)\right|^2 \frac{1}{T^2}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right) + \frac{1}{T}\left|\widehat{w}(\nu)\right|^2\left(\mathrm{E}\left[\left|Z^{\mathrm{A}}\right|^2\right] - \left|\mathrm{E}\left[Z^{\mathrm{A}}\right]\right|^2\left|\phi_{Z^{\mathrm{P}}}(2\pi\nu)\right|^2\right).$$

EXAMPLE 7.6: Pulse interval and amplitude modulation - DPIAM. With i.i.d. amplitudes

$$X(t) = \sum_{n\in\mathbb{Z}} Z_n w\left(t - T_n\right),$$

i.e., equation (2.7) with

- $A_n := 1$, $n \in \mathbb{Z}$;
- $\{T_n\}_{n\in\mathbb{Z}}$ is a renewal process.
- $h(t, Z) := Zw(t)$;

When the amplitudes are correlated we have

$$X(t) = \sum_{n\in\mathbb{Z}} A_n w(t - T_n)$$

i.e., equation (2.7) with

- $\{T_n\}_{n\in\mathbb{Z}}$ is a renewal process.
- $h(t, Z) := w(t)$;

Power Spectrum

The filtering function is now (i.i.d. amplitudes)

$$h(t, Z) = Zw(t)$$

and the basic point process a renewal one. The spectrum of a DPIAM is

$$S_X(\nu) = |\widehat{w}(\nu)|^2 |\mathrm{E}[Z]|^2 S_N(\nu) + \lambda |\widehat{w}(\nu)|^2 \mathrm{Var}(Z)$$

where S_N is given by (5.10).

For instance, we can consider the case of a DPIAM for the transmission of i.i.d. symbols from an alphabet with M values, uniformly distributed. In such a case, Z is uniformly distributed with values in $\{1, 2, \ldots, M-1, M\}$, and the basic point process $\{T_n\}_{n\in\mathbb{Z}}$ is a discrete renewal process with inter-arrivals uniformly taking values in $\{1, 2, \ldots, M-1, M\}$. Therefore, we have a DPIM signal as in Example 7.4 and a PAM with i.i.d. amplitudes. In particular,

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\frac{M+1}{2}, \quad \mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = |\widehat{w}(\nu)|^2 \frac{M(M+1)(2M+1)}{6}$$

Hence, the power spectrum of a DPIAM signal for the transmission of i.i.d. symbols from an alphabet with M values, uniformly distributed, is given by

$$S_X(\nu) = |\widehat{w}(\nu)|^2 \left(\frac{M+1}{2}\right)^2 S_N(\nu) + \lambda |\widehat{w}(\nu)|^2 \left(\frac{M(M+1)(2M+1)}{6} - \left(\frac{M+1}{2}\right)^2\right),$$

where the characteristic function of the inter-arrival times, appearing in the expression of S_N (5.10), is

$$\phi_S(2\pi\nu) = \frac{1}{M} e^{i\pi\nu TM} \frac{\sin(\pi\nu T(M+1))}{\sin(\pi\nu T)}.$$

———

7.5 Time-Hopping Signals

General time-hopping (TH) signals are characterized by a deterministic periodic pattern that allows for multi-user detection. Such a pattern, commonly called signature, consists of a sequence of pulses positioned with respect to a regular grid. The temporal structure of such a signal is described with a point process N having random times

$$T_n = T_n^{\text{TH}} + \sum_{l=0}^{L_c-1} (lT + c_l T_c) \ , \tag{7.8}$$

where

- $\{c_n\}$, $n = 0, \ldots, L_c$, is the deterministic sequence characterizing the L_c-periodic pattern;
- T is the period of the regular grid ($T_c < T$);
- T_n^{TH} is a sequence of random times that depends on the type of temporal modulation (usually PPM).

Figure 7.3 depicts a TH signal with a signature of three pulses. We remark that the sequence of random times (7.8) corresponds to a cluster point process Cox and Isham [1980] with seeds T_n^{TH} and deterministic clusters $\sum_{l=0}^{L_c-1} (lT + c_l T_c)$.

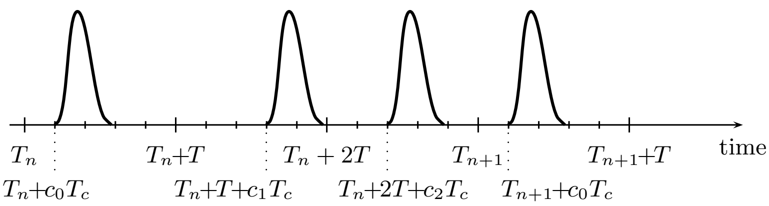


Figure 7.3: Time-hopping signal with signature $\{c_0, c_1, c_2\}$, where $T_c = T/4$, $c_0 = 1$, $c_1 = 3$, and $c_2 = 3$.

A general model for TH signals is given by Win [2002]

$$X(t) = \sum_{k=-\infty}^{\infty} A_n^{\text{TH}} w(t - T_n) = \sum_{k=-\infty}^{\infty} A_n^{\text{TH}} \sum_{l=0}^{L_c-1} w\left(t - T_n^{\text{TH}} - lT - c_l T_c\right) \ , \tag{7.9}$$

where $\{A_n^{\text{TH}}\}_{n\in\mathbb{Z}}$ is a sequence of random variables implementing amplitude modulation, if present. Such model can be seen as the convolution of the pulse shape w with the point process (7.8), or the convolution of the pulse shape

$$\sum_{l=0}^{L_c-1} w(t - lT - c_l T_c) \ ,$$

with the point process having points

$$T_n = T_n^{\text{TH}}$$

We shall adopt the latter modeling approach, being more practical in view of the introduction of additional random quantities. However, in both cases we have a shot noise. In particular, for the latter case (the one we will consider in the following), we have the shot noise of equation (7.2) with random points $T_n = T_n^{\text{TH}}$, $n \in \mathbb{Z}$, and

- $A_n := 1$, $n \in \mathbb{Z}$;
- $Z := A^{\text{TH}}$;
- $h(t, Z) := A^{\text{TH}} \sum_{l=0}^{L_c-1} w(t - lT - c_l T_c)$;

if the sequence $\{A_n^{\text{TH}}\}_{n\in\mathbb{Z}}$ is i.i.d., or

- $A_n := A_n^{\text{TH}}$, $n \in \mathbb{Z}$;
- $h(t, Z) := \sum_{l=0}^{L_c-1} w(t - lT - c_l T_c)$;

if the sequence $\{A_n^{\text{TH}}\}_{n\in\mathbb{Z}}$ is correlated.

We now provide sime examples of common pulse modulations with time-hopping signaling.

EXAMPLE 7.7: Time-hopping and PPM/PAM. Call $\{Z_n^{\text{P}}\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the position modulation and $\{Z_n^{\text{A}}\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the amplitude modulation (for the sake of simplicity we consider the case of i.i.d. amplitudes; the correlated case straightforwardly follows from the previous examples).

The temporal structure is now

$$T_n = U + nL_c T\,, \tag{7.10}$$

where U is a $[0, L_c T]$ uniformly distributed random variable. A time-hopping pulse and amplitude modulated signal can be then written as

$$X(t) = \sum_{k=-\infty}^{\infty} Z_n^{\text{A}} \sum_{l=0}^{L_c-1} w(t - U - nL_c T - Z_n^{\text{P}} - lT - c_l T_c)\ . \tag{7.11}$$

Such an expression corresponds to the model presented by Win [2002]. Again, we have the shot noise with random excitation (7.2) where the basic point process has clustered random times given by (7.10) and

- $A_n := 1$, $n \in \mathbb{Z}$;
- $Z = (Z^{\text{A}}, Z^{\text{P}})$;
- $h(t, Z) := Z^{\text{A}} \sum_{l=0}^{L_c-1} w(t - Z^{\text{P}} - lT - c_l T_c)$.

EXAMPLE 7.8: Time-hopping PPM/PAM signal with jitter and thinning. Consider the case of a time-hopping with pulse position and amplitude modulation that is affected by i.i.d. clock jitter and i.i.d. random losses. Following examples 2.2 and 2.4, jitter and thinning are introduced using the marks (we remark that the case of correlated losses can modeled through the correlated sequence $\{A_n\}_{n\in\mathbb{Z}}$, with values in $\{0, 1\}$).

More precisely, we introduce the i.i.d. sequences of i.i.d. L_c-uplets

$$\left\{\left(Z_{0;n}^{\text{J}}, \ldots, Z_{L_c-1;n}^{\text{J}}\right)\right\}_{n\in\mathbb{Z}}\,, \quad \text{and} \quad \left\{\left(Z_{0;n}^{\text{L}}, \ldots, Z_{L_c-1;n}^{\text{L}}\right)\right\}_{n\in\mathbb{Z}}\,,$$

to model, respectively, the random displacements and the random losses (Z^{L} is a binary sequence). We remark that, due to the clustered nature of the random times, the introduction of L_c-uplets is necessary in order to consider jitter and thinning of each pulse of the signature.

Then, in the presence of i.i.d. jitter and i.i.d. random losses, the time-hopping pulse position and amplitude modulated signal of equation (7.11) reads

$$X(t) = \sum_{n \in \mathbb{Z}} Z_n^{\text{A}} \sum_{l=0}^{L_c-1} Z_{l;n}^{\text{L}} w\left(t - U - nL_cT - Z_n^{\text{P}} - lT - c_lT_c - Z_{l;n}^{\text{J}}\right) . \tag{7.12}$$

Again, it is a shot noise with random excitation, where the sequence of marks is

$$Z_n = \left(Z_n^{\text{A}}, Z_n^{\text{P}}, \left(Z_{0;n}^{\text{J}}, \ldots, Z_{L_c-1;n}^{\text{J}}\right), \left(Z_{0;n}^{\text{L}}, \ldots, Z_{L_c-1;n}^{\text{L}}\right)\right) ,$$

and the impulse response reads

$$h(t, Z) = Z^{\text{A}} \sum_{l=0}^{L_c-1} Z_l^{\text{L}} w\left(t - Z^{\text{P}} - Z_l^{\text{J}} - lT - c_lT_c\right) ,$$

with

$$T_n = U + nL_cT .$$

Note that expression (7.12) is an extension of the model presented in [Win, 2002] that takes into account random losses of the pulses as well as a general jitter. Moreover, with respect to the work of [Win, 2002], our approach allows to complexify the model *ad libitum*, still providing explicit expressions of the power spectra.

Power Spectrum

We consider the time-hopping model of equation (7.9) and we call with $\mu_N(d\nu)$ the Bartlett power spectrum the basic point process $\{T_n\}_{n\in\mathbb{Z}}$.

The general filtering function, when the amplitudes are i.i.d., is

$$h(t, Z) := Z \sum_{l=0}^{L_c-1} w\left(t - lT - c_lT_c\right) ,$$

($A^{\text{TH}} := Z$). Therefore

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\,\mathrm{E}[Z] \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_lT_c)} ,$$

$$\mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = \left|\widehat{w}(\nu)\right|^2 \mathrm{E}\left[|Z|^2\right] \left|\sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_lT_c)}\right|^2 .$$

The spectrum is then straightforwardly obtained by applying equation (7.3), as we show in the following example.

Example 7.9: Spectrum of a time hopping PPM/PAM signal. As in Example 7.7, we call $\{Z_n^{\text{P}}\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the position modulation and $\{Z_n^{\text{A}}\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the i.i.d. amplitude modulation (the correlated case can be straightforwardly taken into account).

Now $T_n = U + nL_cT$, where U is $[0, L_cT]$-uniformly distributed. Therefore, its Bartlett pseudo spectral density is (see Example 3.1)

$$S_N = \frac{1}{(L_cT)^2} \sum_{n \neq 0} \delta \left(\nu - \frac{n}{L_cT} \right)$$

Notice that the intensity λ is now $1/L_cT$.

The filtering function is $h(t, Z) := Z^{\text{A}} \sum_{l,j=0}^{L_c-1} w(t - Z^{\text{P}} - lT - c_lT_c)$, $(Z = (Z^{\text{A}}, Z^{\text{P}}))$, and therefore

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\, \mathrm{E}\left[Z^{\text{A}}\right] \phi_{Z^{\text{P}}}(-2\pi\nu) \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)},$$

$$\mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = \left|\widehat{h}(\nu)\right|^2 \mathrm{E}\left[\left|Z^{\text{A}}\right|^2\right] \left|\sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)}\right|^2$$

The power spectral pseudo density of a time-hopping PPM/PAM signal is then

$$S_X(\nu) = |\widehat{w}(\nu)|^2 \left|\mathrm{E}\left[Z^{\text{A}}\right]\right|^2 \left|\phi_{Z^{\text{P}}}(2\pi\nu)\right|^2 \left|\sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)}\right|^2 \frac{1}{(L_cT)^2} \sum_{n \neq 0} \delta\left(\nu - \frac{n}{L_cT}\right)$$
$$+ \frac{1}{L_cT} |\widehat{w}(\nu)|^2 \left|\sum_{l,j=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)}\right|^2 \left(\mathrm{E}\left[\left|Z^{\text{A}}\right|^2\right] - \left|\mathrm{E}\left[Z^{\text{A}}\right]\right|^2 \left|\phi_{Z^{\text{P}}}(2\pi\nu)\right|^2\right).$$

EXAMPLE 7.10: Spectrum of a time-hopping PPM/PAM signal with jitter and thinning. Following Example 7.8, the filtering function is now

$$h(t, Z) = Z^{\text{A}} \sum_{l=0}^{L_c-1} Z_l^{\text{L}} w\left(t - Z^{\text{P}} - Z_l^{\text{J}} - lT - c_lT_c\right)$$

where Z^{A} models PAM random amplitudes, Z^{L} models the random losses, Z^{P} models PPM random positions and Z^{J} models the jitter. Hence,

$$\mathrm{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\, \mathrm{E}\left[Z^{\text{A}}\right] \mathrm{E}\left[Z^{\text{L}}\right] \phi_{Z^{\text{P}}}(-2\pi\nu)\, \phi_{Z^{\text{J}}}(-2\pi\nu) \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)}$$

$$\mathrm{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = |\widehat{w}(\nu)|^2\, \mathrm{E}\left[\left|Z^{\text{A}}\right|^2\right]$$
$$\left(L_c \mathrm{E}\left[\left|Z^{\text{L}}\right|^2\right] + \left|\mathrm{E}\left[Z^{\text{L}}\right]\right|^2 \left|\phi_{Z^{\text{J}}}(2\pi\nu)\right|^2 \left(\left|\sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT + c_lT_c)}\right|^2 - L_c\right)\right)$$

The power spectral pseudo density is then

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 \frac{1}{(\mathcal{L}T)^2}$$
$$\Bigg(|\mathrm{E}[Z^{\text{A}}]|^2 |\mathrm{E}[Z^{\text{L}}]|^2 |\phi_{Z^{\text{P}}}(2\pi\nu)|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \left| \sum_{l=0}^{\mathcal{L}-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 \sum_{n\neq 0} \delta\left(\nu - \frac{n}{\mathcal{L}T}\right)$$
$$+ L_c \mathrm{E}\left[|Z^{\text{A}}|^2\right] \left(\mathrm{E}\left[|Z^{\text{L}}|^2\right] - |\mathrm{E}[Z^{\text{L}}]|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \right)$$
$$+ |\mathrm{E}[Z^{\text{L}}]|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \left| \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 \left(\mathrm{E}\left[|Z^{\text{A}}|^2\right] - |\mathrm{E}[Z^{\text{A}}]|^2 |\phi_{Z^{\text{P}}}(2\pi\nu)|^2 \right) \Bigg).$$

7.6 Direct-Sequence Signals

As for time-hopping, a direct-sequence signal allows for multi-user detection. It consists of multiplying a stream of pulses by a periodic deterministic sequence of $+1$ and -1 (the signature). It can be generally modeled as

$$X(t) = \sum_{n\in\mathbb{N}} \sum_{k=0}^{L_d-1} a_k^{\text{DS}} w\left(t - T_{nL_d+k}\right) \tag{7.13}$$

where

- L_d is the period of the signature sequence;
- a_k^{DS}, $k = 0, \ldots, L_d - 1$, is the signature sequence, where $a^{\text{DS}} \in \{+1, -1\}$.

Figure 7.4 depicts a direct-sequence signal.

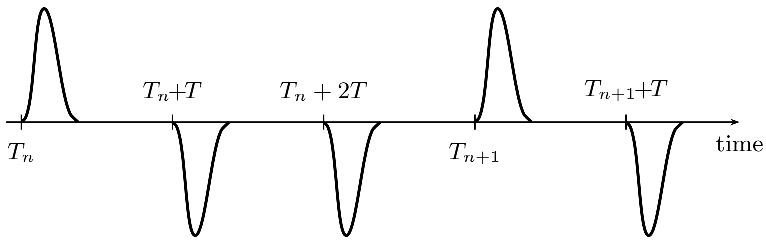


Figure 7.4: Direct-sequence signal with signature $\{a_0, a_1, a_2\}$, where $a_0 = 1$, $a_1 = -1$, and $a_2 = -1$.

Example 7.11: Direct-sequence TH PPM/PAM signal in the presence of jitter and random losses. Modeling a direct-sequence time-hopping with PPM and PAM in the presence of jitter and random losses it is a matter of combining the examples we have presented so far.

As in Example 7.8 we introduce the i.i.d. sequences of i.i.d. L-uplets, where $L = \max(L_c, L_d)$,

$$\left\{\left(Z^{\mathrm{J}}_{0;n}, \ldots, Z^{\mathrm{J}}_{L-1;n}\right)\right\}_{n\in\mathbb{Z}} \text{ and } \left\{\left(Z^{\mathrm{L}}_{0;n}, \ldots, Z^{\mathrm{L}}_{L-1;n}\right)\right\}_{n\in\mathbb{Z}},$$

to model, respectively, the random displacements and the random losses. Call $\{Z^{\mathrm{P}}_n\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the position modulation and $\{Z^{\mathrm{A}}_n\}_{n\in\mathbb{Z}}$ the i.i.d. sequence modeling the amplitude modulation (as in Example 7.5).

We assume that the period L_d of the direct-sequence signature and the period L_c of the time-hopping signature are multiples of each other. We distinguish three cases:

1. $L_c = L_d$

$$X(t) = \sum_{n\in\mathbb{N}} Z^{\mathrm{A}}_n \sum_{k=0}^{L_d-1} Z^{\mathrm{L}}_{k;n} a^{\mathrm{DS}}_k w\left(t - U - nL_cT - Z^{\mathrm{P}}_n - kT - c_kT_c - Z^{\mathrm{J}}_{k;n}\right) \tag{7.14}$$

2. $L_c > L_d$

$$X(t) = \sum_{n\in\mathbb{N}} Z^{\mathrm{A}}_n \sum_{l=0}^{\frac{L_c}{L_d}-1} \sum_{k=0}^{L_d-1} Z^{\mathrm{L}}_{lL_d+k;n} a^{\mathrm{DS}}_k w\left(t - U - nL_cT - Z^{\mathrm{P}}_n - (lL_d+k)T - c_{lL_d+k}T_c - Z^{\mathrm{J}}_{lL_d+k;n}\right) \tag{7.15}$$

3. $L_c < L_d$

$$X(t) = \sum_{n\in\mathbb{N}} Z^{\mathrm{A}}_n \sum_{k=0}^{\frac{L_d}{L_c}-1} \sum_{l=0}^{L_c-1} Z^{\mathrm{L}}_{kL_c+l;n} a^{\mathrm{DS}}_{kL_c+l} w\left(t - U - nL_dT - Z^{\mathrm{P}}_n - (kL_c+l)T - c_kT_c - Z^{\mathrm{J}}_{kL_c+l;n}\right) \tag{7.16}$$

All three cases correspond to the shot noise with random excitation (7.2) where, by identification, we have

- $A_n := 1$, $n \in \mathbb{Z}$;
- $Z := \left(Z^{\mathrm{A}}, Z^{\mathrm{P}}, \left(Z^{\mathrm{J}}_0, \ldots, Z^{\mathrm{J}}_{L_c-1}\right), \left(Z^{\mathrm{L}}_0, \ldots, Z^{\mathrm{L}}_{L_c-1}\right)\right)$;
- if $L_c = L_d$:
  $$h(t, Z) := Z^{\mathrm{A}} \sum_{k=0}^{L_d-1} Z^{\mathrm{L}}_k a^{\mathrm{DS}}_k w\left(t - Z^{\mathrm{P}} - kT - c_kT_c - Z^{\mathrm{J}}_k\right);$$
- if $L_c > L_d$:
  $$h(t, Z) := Z^{\mathrm{A}} \sum_{l=0}^{\frac{L_c}{L_d}-1} \sum_{k=0}^{L_d-1} Z^{\mathrm{L}}_{lL_d+k} a^{\mathrm{DS}}_k w\left(t - Z^{\mathrm{P}} - (lL_d+k)T - c_{lL_d+k}T_c - Z^{\mathrm{J}}_{lL_d+k}\right);$$
- if $L_c < L_d$:
  $$h(t, Z) := Z^{\mathrm{A}} \sum_{k=0}^{\frac{L_d}{L_c}-1} \sum_{l=0}^{L_c-1} Z^{\mathrm{L}}_{kL_c+l} a^{\mathrm{DS}}_{kL_c+l} w\left(t - Z^{\mathrm{P}} - (kL_c+l)T - c_kT_c - Z^{\mathrm{J}}_{kL_c+l}\right)$$

Power Spectrum

We consider the spectrum of direct-sequence time-hopping PPM/PAM signals in the presence of jitter and random losses, as in Example 7.11. Without loss of generality we assume that the direct-sequence and the time hopping signatures have the same period, *i.e.*, $L_c = L_d$ (the other cases easily follows using the corresponding equations in Example 7.11). We call L such a common period.

The filtering function is now given by

$$h(t, Z) = Z^{\text{A}} \sum_{k=0}^{L-1} Z_k^{\text{L}} a_k^{\text{DS}} w\left(t - Z^{\text{P}} - kT - c_k T_c - Z_k^{\text{J}}\right)$$

where a^{DS} represent the direct-sequence signature, Z^{A} models PAM random amplitudes, Z^{L} models the random losses, Z^{P} models PPM random positions and Z^{J} models the jitter. Therefore

$$\text{E}\left[\widehat{h}(\nu, Z)\right] = \widehat{w}(\nu)\, \text{E}\left[Z^{\text{A}}\right] \text{E}\left[Z^{\text{L}}\right] \phi_{Z^{\text{P}}}(-2\pi\nu)\, \phi_{Z^{\text{J}}}(-2\pi\nu) \sum_{l=0}^{L-1} a_l^{\text{DS}} e^{-i2\pi\nu(lT + c_l T_c)}$$

$$\text{E}\left[\left|\widehat{h}(\nu, Z)\right|^2\right] = |\widehat{w}(\nu)|^2\, \text{E}\left[|Z^{\text{A}}|^2\right] \left(L\text{E}\left[|Z^{\text{L}}|^2\right] + \left|\text{E}\left[Z^{\text{L}}\right]\right|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \left(\left|\sum_{l=0}^{L-1} a_l^{\text{DS}} e^{-i2\pi\nu(lT + c_l T_c)}\right|^2 - L\right)\right)$$

The basic point process has spectrum (see Example 7.7)

$$S_N(\nu) = \frac{1}{(LT)^2} \sum_{n \neq 0} \delta\left(\nu - \frac{n}{LT}\right)$$

Finally, the power spectral pseudo density of a direct-sequence time-hopping PPM/PAM signals in the presence of jitter and random losses (7.14) is given by

$$\begin{aligned} S_X(\nu) = {} & |\widehat{w}(\nu)|^2 \left|\text{E}\left[Z^{\text{A}}\right]\right|^2 \left|\text{E}\left[Z^{\text{L}}\right]\right|^2 |\phi_{Z^{\text{P}}}(2\pi\nu)|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \\ & \left|\sum_{l=0}^{L-1} a_l^{\text{DS}} e^{-i2\pi\nu(lT + c_l T_c)}\right|^2 \frac{1}{(LT)^2} \sum_{n \neq 0} \delta\left(\nu - \frac{n}{LT}\right) \\ & + \frac{1}{LT} |\widehat{w}(\nu)|^2 L\text{E}\left[|Z^{\text{A}}|^2\right] \left(\text{E}\left[|Z^{\text{L}}|^2\right] - \left|\text{E}\left[Z^{\text{L}}\right]\right|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2\right) \\ & + \frac{1}{LT} |\widehat{w}(\nu)|^2 \left|\text{E}\left[Z^{\text{L}}\right]\right|^2 |\phi_{Z^{\text{J}}}(2\pi\nu)|^2 \left|\sum_{l=0}^{L-1} a_l^{\text{DS}} e^{-i2\pi\nu(lT + c_l T_c)}\right|^2 \left(\text{E}\left[|Z^{\text{A}}|^2\right] - \left|\text{E}\left[Z^{\text{A}}\right]\right|^2 |\phi_{Z^{\text{P}}}(2\pi\nu)|^2\right) \end{aligned}$$

We can remark that the different features of the model (type of direct-sequence, characteristic of the TH, type of modulation, as well as the characteristic of the jitter and the random losses) appear clearly and separately.

Spectrum of direct-sequence TH PPM/PAM signals is, as far as we know, an original contribution. As already mentioned, a more substantial contribution is given by the method and the tools we have use to obtain such formula. We emphasize the generality of the result that presents a generic direct-sequence and TH, a generic jitter and generic losses. Moreover, different pulse modulations can be taken into account by simply replacing the spectrum S_N and the random pulse function.

Chapter 8

Multipath Fading Channels

Summary: In this chapter we address the problem of modeling received pulse trains in a multipath fading channel and of computing their exact spectrum. This is an example of practical interest of a highly complex signal obtained from a random spike field.

Our contribution: We propose a model based on a shot noise with random excitation and modulated basic point process. Such model is very general, simple and tractable and it allows to account for various phenomena that affect the transmission. We then give the exact spectrum of the output of such a model. Spectral formula of specific configurations are then derived from a singular general formula, where the various features of the channel and the pulse transmission appear explicitly.

As discussed in the previous chapter, spectrum of UWB signals has benefit of several contributions. On the contrary, results on the exact spectrum computation of the output of a general multipath fading channel, when fed by an UWB signal or more generally a pulse train, are not available in the scientific literature.

Here, we address two main problems. The first one is to derive a very general, yet tractable, model for the output of a multipath fading channel when the input is a general pulse train or, as a special case, a UWB signal. The second is to provide the exact spectrum for such a model.

We propose a model based on point processes, where

- the positions of the pulses and of their replica due to multipaths are realizations of a point processes;
- the pulse shape is a random function;
- fluctuations of pulse amplitudes are taken into account through the modulation with a w.s.s. process;
- the different multipath components are attenuated by random functions.

Such a model is very general, simple and tractable, and, from the point process perspective, it corresponds to a shot noise with random excitation, where the basic point process is modulated by a w.s.s. process. It allows to finely account for various phenomena that affects the transmission, such as jitter, attenuation, losses and distortion of the pulses. As a special case, it provides the classical double Poisson model of Saleh and Valenzuela [1987].

Within the framework of our model, the power spectra of received pulse trains in multipath fading channels can then be derived by exploiting the results we have presented in Chapter 4. The exact power spectrum formula we obtain is easy to understand since the contributions corresponding to various features such as pulse modulation, multipath repetitions, fading effects as well as other random effects, *appear clearly and separately* in the power spectrum expression.

Power spectrum of a pulse train through a general multipath fading channel is, to the best of our knowledge, a novel result.

Here, we focus on a one-dimensional model of signals taking real values, which is sufficient for our purpose. Extension to signals with complex values is straightforward while the extension to multi dimensions can be made following the general definitions of spike fields and related processes given in Chapter 1.

8.1 The Model

We consider a situation where a pulse train is transmitted through a multipath fading channel, as depicted in Figure 8.1. We will develop the model step by step, modularly adding the type of pulse modulation, the fading, the multipath repetitions and additional random effects.

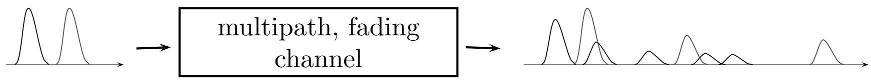


Figure 8.1: Pulse transmission through a fading multipath channel.

8.1.1 Pulse Modulation

As described in Chapter 7, pulse modulated signals, and more generally, UWB signals, can be modeled as a shot noise with random excitation.

$$X(t) = \int_{\mathbb{R}} w(t-s,z)\,\bar{N}(ds \times dz) = \sum_{n \in \mathbb{Z}} w(t - T_n, Z_n) \ , \tag{8.1}$$

where

- $w(s,z) : \mathbb{R} \times E \to \mathbb{R}$ is a function that depends on a random parameter z taking values over E, where $w(t,\cdot)$ represents the pulse shape;
- $\bar{N}$ is a marked point process with underlying point process N that is simple, locally finite and stationary, and i.i.i.d. marks $\{Z_n\}_{n\in\mathbb{Z}}$ taking values over E and with common distribution Q_Z. The marks allows to account for random effects on the pulses, such as random amplitude, jitter and thinning.

We refer to Chapter 7 for examples of specific modulations.

8.1.2 Fading

We consider fading as random modifications of the pulse introduced by the channel, such as attenuation and/or distortion. We shall consider the fading process to be stationary, at least in wide sense.

Pulse Attenuation

We model the random attenuation through the multiplication of the train of pulses with a correlated w.s.s. stochastic process $\{V(t)\}_{t\in\mathbb{R}}$, independent of the train itself. Equivalently, considering a relatively short duration of the pulses, we can multiply the point process N by the stochastic process $\{V(t)\}_{t\in\mathbb{R}}$ and then convolute the so obtained modulated point process with the pulse shape. Using the latter approach, the effect of the random attenuation on the input is

$$X(t) = \int_{\mathbb{R}\times R} w(t-s,z)\,V(s)\,\bar{N}(ds\times dz) = \sum_{n\in\mathbb{Z}} w(t-T_n,Z_n)\,V(T_n)\ .$$

We remark that a situation in which the attenuation of the pulses are i.i.d. is better described using the i.i.d. marks $\{Z\}_{n\in\mathbb{Z}}$ rather than the w.s.s. process $\{V(t)\}_{t\in\mathbb{R}}$ (see for instance pulse amplitude modulation described in Section 7.2).

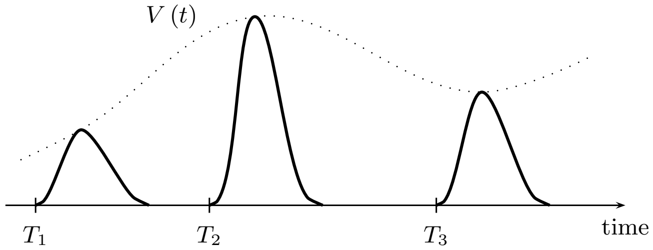


Figure 8.2: Fading on a pulse modulation.

Pulse Distortion

Random i.i.d. distortions of the pulses can be modelled through

- a random parameter characterizing the pulse shape, *e.g.*, random amplitude or width;
- the convolution of the pulse with a random function.

In both cases, the randomness is captured through the i.i.d. marks $\{Z_n\}_{n\in\mathbb{Z}}$ of the marked point process.

Random Parameters. An easy example is provided by a pulse with an amplitude that oscillates around a certain value. We can express such a situation as

$$w(t,Z) = (1+Z)\,w(t)\,,\quad t\in\mathbb{R}\,,$$

where $\mathrm{E}[Z]=0$.

Convolution with a Random Function. We consider $w(t,Z)$, $t\in\mathbb{R}$ as the convolution of the deterministic pulse function and some distorting random function, *i.e.*,

$$w(t,Z) = (w(\cdot) * \xi(\cdot,Z))(t)\,,\quad t\in\mathbb{R}\,.$$

Then

$$\widehat{w}(\nu,Z) = \widehat{w}(\nu)\,\widehat{\xi}(\nu,Z)\,,\quad \nu\in\mathbb{R}\,.$$

For example, the distortion can be accounted via the convolution of the pulse with a function randomly chosen among a certain set of functions, *e.g.*, a set composed of different square pulses and/or triangular pulses,

$$\xi(t,Z) = \sum_{k=1}^{K} \xi_k(t)\,1_{\{Z=k\}}\,,$$

where now Z is a random variables taking values over $\{1,\ldots,K\}$.

8.1.3 Multipaths

The multipath effects consist of attenuated repetitions of the transmitted pulse due to reflections by surrounding objects. It has been shown that reflected pulses arrive in clusters [Saleh and Valenzuela, 1987]. This suggests a model where the reflected pulses are divided into *principal reflections* and *secondary scattering*. To each pulse corresponds a sequence of repetitions due to principal reflections, that we shall call *principal pulse repetitions*. Then for each principal repetition (also including the pulse itself) we have a sequence of additional repetitions due to the secondary scattering, that we shall call *secondary pulse repetitions*. Principal repetitions have lower arrival rate and amplitudes attenuating less rapidly than the secondary repetitions. Figure 8.3 depicts a schematic representation of multipath repetitions presented in [Saleh and Valenzuela, 1987].

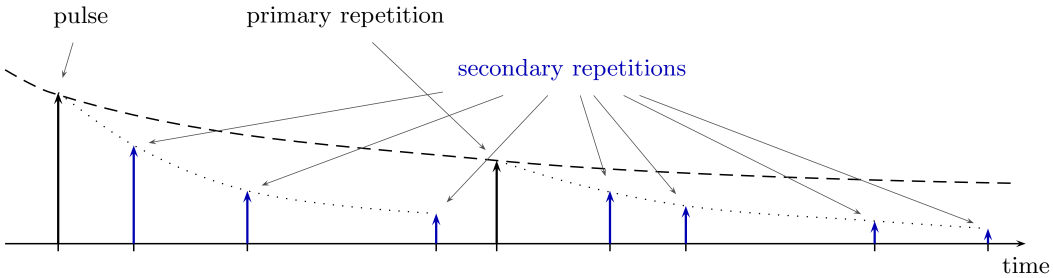


Figure 8.3: Clusters of reflected pulses.

We will first focus the principal reflections and then introduce the secondary scattering.

Principal Reflection Multipaths

The modelling of the principal reflection multipaths requires the introduction of

- a sequence of collections of random points $\{T^{\mathrm{p}}_{n;l}, l \geq 1\}$, $n \in \mathbb{Z}$, where $\{T^{\mathrm{p}}_{n;l}\}_{l\geq 1}$ describes the relative positions of the primary repetitions of the n-th pulse;
- a function $g^{\mathrm{p}}(t)$, $t \in \mathbb{R}_+$ describing the attenuation of the repeated pulses.

Commonly, $\{T^{\mathrm{p}}_{n;l}\}_{l\geq 1, n\in\mathbb{Z}}$ are called *propagation delays*, and $\{g^{\mathrm{p}}(T^{\mathrm{p}}_l)\}_{l\geq 1}$ *gains*. We can also take into account truncated sequence of random times $\{T^{\mathrm{p}}_1, \ldots, T^{\mathrm{p}}_L\}$, where L is an integer-valued random variable, independent of the random points (see Example 8.7). However, such a situation can be approximated by an infinite sequence of random times attenuated by a rapidly decaying function.

Using the propagation delays and gains, a general sequence of pulses (depending on a random parameter) and their attenuated repetitions can be written as

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - T_n, Z_n) + \sum_{n\in\mathbb{Z}} \sum_{l\geq 1} w(t - T_n - T^{\mathrm{p}}_l, Z_n)\, g^{\mathrm{p}}(T^{\mathrm{p}}_l) ,$$

where $\{T_n\}_{n\in\mathbb{N}}$ is the sequence of the random positions of the pulses at the input of the channel.

The model can be straightforwardly complexified by assuming that the attenuating function g^{p} depends on a random parameter. More precisely, we define

- $g^{\mathrm{p}}(s, z) : \mathbb{R}_+ \times E^{\mathrm{p}} \to \mathbb{R}$ to be a function depending on a random parameter that takes values over some space E^{p} (it determines the attenuation of the pulses corresponding to the principal repetitions);

- $\bar{N}_n^{\mathrm{p}} = \{(T_{n;l}^{\mathrm{p}}, Z_{n;l}^{\mathrm{p}}), l \geq 1\}$, $n \in \mathbb{Z}$, to be an i.i.d. sequence of marked point processes, where, for each n, the marked point process $\bar{N}_n^{\mathrm{p}}$ has a basic point process $N_n^{\mathrm{p}} = \{T_{n;l}^{\mathrm{p}}, l \geq 1\}$ and i.i.d. marks $\{Z_{n;l}^{\mathrm{p}}\}_{l\geq 1}$ having common distribution $Q_{Z^{\mathrm{p}}}$ (see Definition 1.6); We call λ^{p} the common average intensity of the basic point processes.

Then, the output of the channel reads

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - T_n, Z_n) + \sum_{n\in\mathbb{Z}} \sum_{l\geq 1} w\left(t - T_n - T_{n;l}^{\mathrm{p}}, Z_n\right) g^{\mathrm{p}}\left(T_{n;l}^{\mathrm{p}}, Z_{n;l}^{\mathrm{p}}\right) \tag{8.2}$$

Figure 8.4 depicts two Dirac pulses and the corresponding principal repetitions.

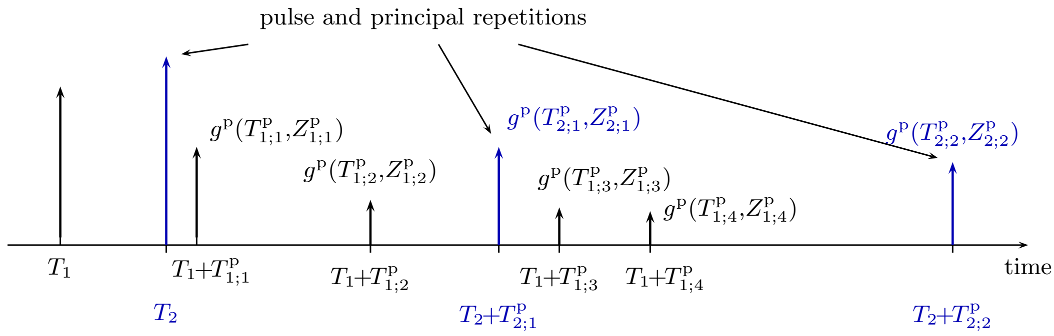


Figure 8.4: Two Dirac pulses (at random times T_1 and T_2) with the corresponding principal repetitions.

Now, by letting $Z_n^{\mathrm{c}} = \left(Z_n, \bar{N}_n^{\mathrm{p}}\right)$, $n \in \mathbb{Z}$, and

$$H(t - T_n, Z_n^{\mathrm{c}}) = w(t - T_n, Z_n) + \sum_{l\geq 1} w\left(t - T_n - T_{n;l}^{\mathrm{p}}, Z_n\right) g^{\mathrm{p}}\left(T_{n;l}^{\mathrm{p}}, Z_{n;l}^{\mathrm{p}}\right), \quad n \in \mathbb{Z}, \tag{8.3}$$

we can interpret expression (8.2) as the convolution of the random impulse response $H(t, z)$ with a marked point process $\bar{N}^{\mathrm{c}}$ with basic random points $\{T_n\}_{n\in\mathbb{Z}}$ and i.i.d. marks $\{Z_n^{\mathrm{c}}\}_{n\in\mathbb{Z}}$ taking values over some space E^{c}, *i.e.*,

$$X(t) = \sum_{n\in\mathbb{Z}} H(t - T_n, Z_n^{\mathrm{c}}) = \int_{\mathbb{R}\times E^{\mathrm{c}}} H(t - s, z)\, \bar{N}^c(ds \times dz),$$

Hence, a pulse train with multipath repetitions can be modeled as a shot noise with random excitation (see Definition 2.2). Notice that here the marked point process $\bar{N}^{\mathrm{p}}$ plays the role of a *random cluster*. In this context, $\bar{N}^{\mathrm{c}}$ is a *cluster point process* (indeed, cluster point processes can be seen as a particular case of shot noises with random excitation - see Section 2.4).

Principal Reflections and Secondary Scattering Multipaths

We now introduce the repetitions due to secondary scattering multipaths. We follow the same approach adopted for the principal repetitions. However, we remark that secondary scattering multipaths are generated by both the pulses and their primary repetitions. Therefore, in order to introduce the secondary repetitions we associate to each pulse and to each of its primary repetitions a sequence of random points. More precisely, we introduce

- a double sequence of collections of random points

$$\{T^{\mathrm{s}}_{n,l;k}, k \geq 1\}, \quad n \in \mathbb{Z}\,, \quad l \geq 1\,,$$

  where $\{T^{\mathrm{s}}_{n,l;k}\}_{k\geq 1}$ describes the relative positions of the secondary repetitions that where generated by the l-th primary repetition of the n-th pulse; We shall denote with $\{T^{\mathrm{s}}_{n,0;k}\}_{k\geq 1}$ the secondary repetitions directly generated by the n-th pulse;
  The double sequence, including $\{T^{\mathrm{s}}_{n,0;k}\}_{k\geq 1}$, is supposed to be i.i.d. (with respect to both indexes $n \in \mathbb{Z}$ and $l \in \mathbb{N}$);
- a function

$$g^{\mathrm{s}}(s,z) : \mathbb{R}_{+} \times E^{\mathrm{p}} \to \mathbb{R}$$

  that depends on a random parameter taking values over some space E^{s}, where $g^{\mathrm{s}}(s,\cdot)$ represents the attenuation, or path-loss, of the secondary repetitions;
- a double sequence of collections of i.i.d. random variables

$$\{Z^{\mathrm{s}}_{n,l;k}, k \geq 1\}, \quad n \in \mathbb{Z}, \quad l \in \mathbb{N}\,,$$

  where each random variable $Z^{\mathrm{s}}_{n,l;k}$ represent the attenuating function random parameter that is associated to the relative position $T^{\mathrm{s}}_{n,l;k}$ of the secondary repetitions;
  $\{Z^{\mathrm{s}}_{n,l;k}\}_{n\in\mathbb{Z},l\in\mathbb{N},k\geq 1}$ are supposed to be i.i.d. with respect to all three indexes.

If we consider that the input of the channel is a sequence of pulses, each depending on a random parameter Z, the output, with attenuated primary and secondary repetitions, then reads

$$\begin{aligned} X(t) = & \sum_{n\in\mathbb{Z}} w(t - T_n, Z_n) \\ & + \sum_{n\in\mathbb{Z}} \sum_{k\geq 1} w\left(t - T_n - T^{\mathrm{s}}_{n,0;k}, Z_n\right) g^{\mathrm{s}}\left(T^{\mathrm{s}}_{n,0;k}, Z^{\mathrm{s}}_{n,0;k}\right) \\ & + \sum_{n\in\mathbb{Z}} \sum_{l\geq 1} w\left(t - T_n - T^{\mathrm{p}}_{n;l}, Z_n\right) g^{\mathrm{p}}\left(T^{\mathrm{p}}_{n;l}, Z^{\mathrm{p}}_{n;l}\right) \\ & + \sum_{n\in\mathbb{Z}} \sum_{l\geq 1} \sum_{k\geq 1} w\left(t - T_n - T^{\mathrm{p}}_{n;l} - T^{\mathrm{s}}_{n,l;k}, Z_n\right) g^{\mathrm{p}}\left(T^{\mathrm{p}}_{n;l}, Z^{\mathrm{p}}_{n;l}\right) g^{\mathrm{s}}\left(T^{\mathrm{s}}_{n,l;k}, Z^{\mathrm{s}}_{n,l;k}\right)\,. \end{aligned} \tag{8.4}$$

From the top to the bottom,

- the first term corresponds to the sequence of pulses,
- the second term corresponds to the secondary repetition due to the scattering multipaths of each pulse,
- the third term models the primary repetition due to the principal multipaths of each pulse,
- the last term model the secondary repetitions due to the scattering multipaths of each primary repetition.

Figure 8.5 depicts principal and secondary repetitions of a single pulse.

Using the point process formalism, we can define $\bar{N}^{\mathrm{s}}_{n,l} = \{(T^{\mathrm{s}}_{n,l;k}, Z^{\mathrm{s}}_{n,l;k}), k \geq 1\}$ to be a double sequence of marked point process with i.i.i.d. marks (where the double sequence of marked point processes is i.i.d. with respect to both indexes n and l). We call λ^{s} the average intensity of the corresponding basic point processes (λ^{s} is the same for all basic point processes since the

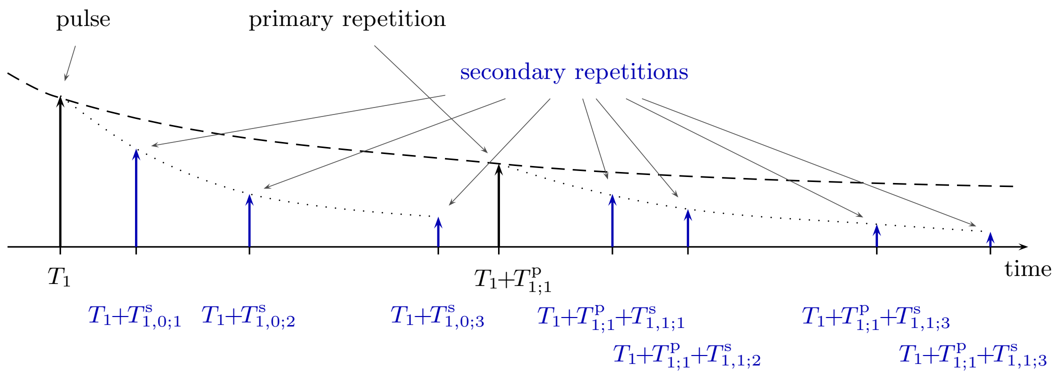


Figure 8.5: One Dirac pulse (at random time T_1) with a principal repetition and the corresponding secondary repetitions.

sequence is i.i.d.). We can then define the random impulse response

$$\begin{aligned} H\left(t-T_n, Z^{\mathrm{c}}_n\right) = w\left(t-T_n, Z_n\right) + \sum_{k\geq 1} w\left(t-T_n-T^{\mathrm{s}}_{n,0;k}, Z_n\right) g^{\mathrm{s}}\left(T^{\mathrm{s}}_{n,0;k}, Z^{\mathrm{s}}_{n,0;k}\right) \\ + \sum_{l\geq 1} w\left(t-T_n-T^{\mathrm{p}}_{n;l}, Z_n\right) g^{\mathrm{p}}\left(T^{\mathrm{p}}_{n;l}, Z^{\mathrm{p}}_{n;l}\right) \\ + \sum_{l\geq 1}\sum_{k\geq 1} w\left(t-T_n-T^{\mathrm{p}}_{n;l}-T^{\mathrm{s}}_{n,l;k}, Z_n\right) g^{\mathrm{p}}\left(T^{\mathrm{p}}_{n;l}, Z^{\mathrm{p}}_{n;l}\right) g^{\mathrm{s}}\left(T^{\mathrm{s}}_{n,l;k}, Z^{\mathrm{s}}_{n,l;k}\right), \quad n\in\mathbb{Z}, \end{aligned} \tag{8.5}$$

where now $Z^{\mathrm{c}}_n = (Z_n, \bar{N}^{\mathrm{p}}_n, \{\bar{N}^{\mathrm{s}}_{n,l}, l\in\mathbb{N}\})$. Then, the output of a channel with principal secondary multipaths can be treated as a shot noise with random excitation

$$\begin{aligned} X(t) &= \sum_{n\in\mathbb{Z}} H\left(t-T_n, Z^{\mathrm{c}}_n\right) \\ &= \int_{\mathbb{R}\times E^{\mathrm{c}}} H(t-s, z)\, \bar{N}^{\mathrm{c}}(ds\times dz), \end{aligned}$$

with random impulse response given by (8.5), and marked point process $\bar{N}^{\mathrm{c}}$ having basic point process $N = \{T_n, n\in\mathbb{Z}\}$ and i.i.i.d. marks

$$Z^{\mathrm{c}}_n = \left(Z_n, \bar{N}^{\mathrm{p}}_n, \left\{\bar{N}^{\mathrm{s}}_{n,l}, l\in\mathbb{N}\right\}\right), \quad n\in\mathbb{Z},$$

taking value over some space E^{c}. We notice that now $\bar{N}^{\mathrm{c}}$ corresponds to a double cluster point process, with nested random clusters $\bar{N}^{\mathrm{p}}$ and $\bar{N}^{\mathrm{s}}$. Obviously, the impulse response (8.3) is a special case of (8.5).

8.1.4 Output of the Multipath Fading Channel

Combining all the effects together, we obtain the output $\{X(t)\}_{t\in\mathbb{R}}$ of a multipath fading channel as a shot noise with random excitation and modulated basic point process.

$$X(t) = \sum_{n\in\mathbb{Z}} H(t - T_n, Z_n^c)\, V(T_n) \tag{8.6a}$$

$$= \int_{\mathbb{R}\times E^c} H(t-s, z)\, V(s)\, \bar{N}^c(ds \times dz) \tag{8.6b}$$

where H is given by (8.3) or by (8.5), depending on whether we are considering the secondary scattering or not. Figure 8.6 depicts an examples of such a signal (for ease of drawing we just consider primary repetitions).

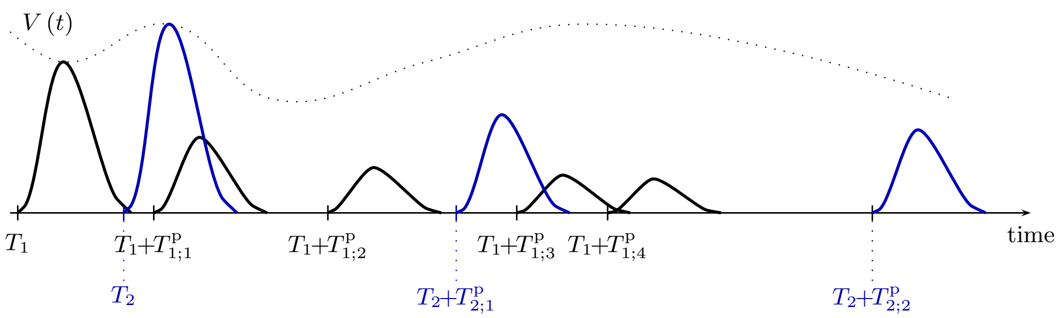


Figure 8.6: Principal reflections and fading for two Dirac pulses at random times T_1 and T_2.

8.2 Power Spectrum

We are interested in computing the power spectrum of a pulse train over a multipath, fading channel, modeled as in (8.6). In the following, we assume that the point processes describing the pulse positions and the primary and secondary repetitions are simple, locally finite, stationary and that they admit a Bartlett spectrum.

As we have discussed in the previous section, the output of the multipath fading channel (8.6) corresponds to a shot noise with random excitation and modulated basic point processes. Therefore, its spectrum can be computed by a straightforward application of Corollary 5.2.1. As we have already remarked concerning the spectrum of UWB signals in Chapter 7, we can avoid to deal with the technicalities related to the domain of definition of the Bartlett spectrum. Indeed, the function w, g^p and g^s are regular bounded functions that rapidly decrease or have finite support, hence contained in the domain of definition of the spectrum (see Chapter 3). Then, the power spectral measure of received pulse trains in multipath, fading channels reads

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{H}(\nu, Z^c)\right]\right|^2 \left(\mu_N * \mu_V(d\nu) + \lambda^2 \mu_V(d\nu) + \mathrm{E}[V]^2 \mu_N(d\nu)\right) + \lambda \mathrm{E}\left[|V(t)|^2\right] \mathrm{Var}\left(\widehat{H}(\nu, Z^c)\right) d\nu\,, \tag{8.7}$$

where, denoting with $\widehat{\ }$ the Fourier transformation,

$$\widehat{H}(\nu, Z^{\mathrm{c}}) = \widehat{w}(\nu, Z)\, F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right) , \tag{8.8}$$

and

$$F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right) = 1 + \sum_{l\geq 1} e^{-i2\pi\nu T_l^{\mathrm{p}}} g^{\mathrm{p}}\left(T_l^{\mathrm{p}}, Z_l^{\mathrm{p}}\right) + \sum_{k\geq 1} e^{-i2\pi\nu T_{0;k}^{\mathrm{s}}} g^{\mathrm{s}}\left(T_{0;k}^{\mathrm{s}}, Z_{0;k}^{\mathrm{s}}\right) + \sum_{l\geq 1}\left(e^{-i2\pi\nu T_l^{\mathrm{p}}} g^{\mathrm{p}}\left(T_l^{\mathrm{p}}, Z_l^{\mathrm{p}}\right) \sum_{k\geq 1} e^{-i2\pi\nu T_{l;k}^{\mathrm{s}}} g^{\mathrm{s}}\left(T_{l;k}^{\mathrm{s}}, Z_{l;k}^{\mathrm{s}}\right)\right) .$$

The above result deserves a few comments:

- the expression of $F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)$ is purposely kept in a general form in order to accomodate various situation of gains and propagation delays;
- in the absence of multipaths $F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right) \equiv 1$ and we obtain the standard form of the spectrum of a shot noise with random excitation and filtering function w.

Concerning the first and second order moments of $\widehat{H}(\nu, Z^{\mathrm{c}})$ we have the following lemma.

Lemma 8.2.1 *Let $\widehat{H}(\nu, Z^{\mathrm{c}})$ be given by (8.8) and call $\mu_{N^{\mathrm{p}}}$ and $\mu_{N^{\mathrm{s}}}$ the Bartlett spectra of the basic point processes of $\bar{N}^{\mathrm{p}}$ and $\bar{N}^{\mathrm{s}}$, respectively. Given a function f, depending on a random parameter with distribution Q, and such that $f \in \mathrm{L}^1_{\mathbb{C}}(\ell \times Q)$, recall the notation*

$$\widehat{\bar{f}}(\nu) = \int_{\mathbb{R}} e^{-i2\pi\nu t}\mathrm{E}\left[f(t, Z)\right] dt$$

(hence $\bar{\ }$ denotes the averaging operation and $\widehat{\ }$ the Fourier transformation). Then

$$\mathrm{E}\left[\widehat{H}(\nu, Z^{\mathrm{c}})\right] = \mathrm{E}\left[\widehat{w}(\nu, Z)\right] \mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right] , \tag{8.9}$$

with

$$\mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right] = 1 + \lambda^{\mathrm{p}}\widehat{\bar{g}}^{\mathrm{p}}(\nu) + \lambda^{\mathrm{s}}\widehat{\bar{g}}^{\mathrm{s}}(\nu) + \lambda^{\mathrm{p}}\lambda^{\mathrm{s}}\widehat{\bar{g}}^{\mathrm{p}}(\nu)\,\widehat{\bar{g}}^{\mathrm{s}}(\nu) , \tag{8.10}$$

and

$$\begin{aligned}\mathrm{Var}\left(\widehat{H}(\nu, Z^{\mathrm{c}})\right) =& \mathrm{E}\left[\left|\widehat{w}(\nu, Z)\right|^2\right] \mathrm{Var}\left(F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right) \\ &+ \mathrm{Var}\left(\widehat{w}(\nu, Z)\right) \left|\mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right]\right|^2\end{aligned} \tag{8.11}$$

with

$$\begin{aligned}\mathrm{Var}\left(F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right) = &\left\{ \left(\int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{p}}(\nu+\upsilon)\right|^2 \mu_{N^{\mathrm{p}}}(d\upsilon) + \lambda^{\mathrm{p}}\int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{p}}(\nu+\upsilon, Z^{\mathrm{p}})\right) d\upsilon\right)\right. \\ &\left.\times\left(\int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{s}}(\nu+\upsilon)\right|^2 \mu_{N^{\mathrm{s}}}(d\upsilon) + \lambda^{\mathrm{s}}\int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{s}}(\nu+\upsilon, Z^{\mathrm{s}})\right) d\upsilon\right)\right\} \\ &+\left|1+\lambda^{\mathrm{s}}\widehat{\bar{g}}^{\mathrm{s}}(\nu)\right|^2 \left(\int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{p}}(\nu+\upsilon)\right|^2 \mu_{N^{\mathrm{p}}}(d\upsilon) + \lambda^{\mathrm{p}}\int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{p}}(\nu+\upsilon, Z^{\mathrm{p}})\right) d\upsilon\right) \\ &+\left|1+\lambda^{\mathrm{p}}\widehat{\bar{g}}^{\mathrm{p}}(\nu)\right|^2 \left(\int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{s}}(\nu+\upsilon)\right|^2 \mu_{N^{\mathrm{s}}}(d\upsilon) + \lambda^{\mathrm{s}}\int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{s}}(\nu+\upsilon, Z^{\mathrm{s}})\right) d\upsilon\right)\end{aligned} \tag{8.12}$$

Proof First of all recall that if $f \in \mathrm{L}^1(\ell \times Q)$, where Q is the distribution of its random parameter, then

$$\int_{\mathbb{R}} e^{-i2\pi\nu t} \mathrm{E}\left[f(t, Z)\right] dt = \mathrm{E}\left[\widehat{f}(\nu, Z)\right] = \widehat{\bar{f}}(\nu) \ .$$

and that, as discussed at the beginning of this section, w, g^{p} and g^{s} are all supposed to be integrable with respect to both their temporal variable and random parameter.

For the first order moment we have

$$\mathrm{E}\left[\widehat{H}(\nu, Z^{\mathrm{c}})\right] = \mathrm{E}\left[\widehat{w}(\nu, Z)\right] \mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right] \ ,$$

where

$$\mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right] = 1 + \lambda^{\mathrm{p}} \widehat{\bar{g}}^{\mathrm{p}}(\nu) + \lambda^{\mathrm{s}} \widehat{\bar{g}}^{\mathrm{s}}(\nu) + \lambda^{\mathrm{p}} \lambda^{\mathrm{s}} \widehat{\bar{g}}^{\mathrm{p}}(\nu) \widehat{\bar{g}}^{\mathrm{s}}(\nu)$$

Concerning the variance, using the conditional variance formula (see for instance [Shiryaev, 1996])

$$\mathrm{Var}\left(f(AB)\right) = \mathrm{E}\left[\mathrm{Var}\left(f(AB) \,|\, \mathcal{F}_A\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[f(AB) \,|\, \mathcal{F}_A\right]\right) \tag{8.13}$$

with $f(AB) = \widehat{w}(\nu, Z) F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)$ and $\mathcal{F}_A = \mathcal{F}_Z$ (where $\mathcal{F}_Z$ is the σ-field generated by Z), we obtain

$$\begin{aligned}\mathrm{Var}\left(\widehat{w}(\nu, Z) F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right) =& \mathrm{E}\left[|\widehat{w}(\nu, Z)|^2\right] \mathrm{Var}\left(F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right) \\ &+ \mathrm{Var}\left(\widehat{w}(\nu, Z)\right) \left|\mathrm{E}\left[F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right]\right|^2 \ .\end{aligned}$$

Call

$$C = \sum_{l \geq 1} e^{-i2\pi\nu T_l^{\mathrm{p}}} g^{\mathrm{p}}\left(T_l^{\mathrm{p}}, Z_l^{\mathrm{p}}\right) \ , \quad C^{(2)} = \sum_{l \geq 1} \left|g^{\mathrm{p}}\left(T_l^{\mathrm{p}}, Z_l^{\mathrm{p}}\right)\right|^2 \ ,$$

and

$$D_l = \sum_{k \geq 1} e^{-i2\pi\nu T_{l;k}^{\mathrm{s}}} g^{\mathrm{s}}\left(T_{l;k}^{\mathrm{s}}, Z_{l;k}^{\mathrm{s}}\right) \ , \quad l \geq 0 \,,$$

and denote with D a process distributed as the common distribution of the sequence $\{D_l\}_{l \in \mathbb{N}}$ (recall that it is an i.i.d. sequence). Then, by applying again the conditional variance formula (8.13) with $f(AB) = F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)$ and $\mathcal{F}_A = \mathcal{F}_{\bar{N}^{\mathrm{p}}}$, we have

$$\mathrm{Var}\left(F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)\right) = \left(1 + \mathrm{E}\left[C^{(2)}\right]\right) \mathrm{Var}(D) + \mathrm{Var}(C) \left|1 + \mathrm{E}[D]\right|^2$$

where

$$\begin{aligned}\mathrm{Var}(C) &= \int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{p}}(\nu + \upsilon)\right|^2 \mu_{N^{\mathrm{p}}}(d\upsilon) + \lambda^{\mathrm{p}} \int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{p}}(\nu + \upsilon, Z^{\mathrm{p}})\right) d\upsilon \,, \\ \mathrm{Var}(D) &= \int_{\mathbb{R}} \left|\widehat{\bar{g}}^{\mathrm{s}}(\nu + \upsilon)\right|^2 \mu_{N^{\mathrm{s}}}(d\upsilon) + \lambda^{\mathrm{s}} \int_{\mathbb{R}} \mathrm{Var}\left(\widehat{g}^{\mathrm{s}}(\nu + \upsilon, Z^{\mathrm{s}})\right) d\upsilon \,,\end{aligned}$$

and

$$\mathrm{E}\left[C^{(2)}\right] = \lambda^{\mathrm{p}} \int_{\mathbb{R}} \mathrm{E}\left[\left|\widehat{g}^{\mathrm{p}}(\upsilon, Z^{\mathrm{p}})\right|^2\right] d\upsilon \,, \quad \mathrm{E}[D] = \lambda^{\mathrm{s}} \widehat{\bar{g}}^{\mathrm{s}}(\nu) \ .$$

Then result (8.12) follows. □

From the results of Lemma 8.2.1, we can remark the modularity of the model: expressions (8.9) and (8.11) evidence the separate contribution of

- $\widehat{w}(\nu, z)$, which takes into account the characteristics of the input pulse train, such as the pulse shape and pulse modulation, the jittering, the thinning and the pulse distortion introduced by the channel;
- $F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)$ which takes into account the type of multipath repetitions;

while expressions (8.10) and (8.12) clearly shows the distinct contributions of the primary and secondary repetitions and their attenuating functions.

8.2.1 Examples

We now give some examples of power spectra of the output of the multipath fading channel. As already remarked, the spectral terms due to pulse modulation, fading, and multipath appear separately and explicitly in the power spectrum expression (8.7). Therefore, we consider them individually and the overall power spectrum expression can be obtained by combining the individual spectral terms.

Pulse Modulation and Related Random Effects

Several examples of pulse modulation with additional random effect were presented in Chapter 7. We consider here an example of a pulse modulation that gives a particular form to the spectrum expression.

Example 8.1: Pulse Position Modulation - PPM. Pulse position modulations where described in Section 7.1. Recall that the basic point process is a regular grid with power spectrum (see Example 3.1)

$$\mu_N(d\nu) = \frac{1}{T^2}\sum_{n\neq 0}\epsilon_{n/T}(d\nu) ,$$

where ϵ_a is the Dirac measure centered at a. In such a case, the convolution of the two measures μ_N and μ_V takes the simple form

$$\mu_N * \mu_V(\nu) = \frac{1}{T^2}\sum_{n\neq 0}\mu_V\left(d\nu - \frac{n}{T}\right) .$$

If in addition we suppose that μ_V admits a density S_V, we have

$$\mu_N * \mu_V(\nu) = \frac{1}{T^2}\sum_{n\neq 0}S_V\left(\nu - \frac{n}{T}\right)d\nu .$$

Pulse Distortion

Example 8.2: Oscillating Amplitude. Consider a pulse amplitude that randomly oscillates around a certain value, *i.e.*,

$$w(t,Z) = w(t)(1+Z) , \quad t\in\mathbb{R},$$

with $\mathrm{E}[Z]=0$. Then $\mathrm{E}[\widehat{w}(\nu,Z)] = \widehat{w}(\nu)$, and $\mathrm{E}\left[|\widehat{w}(\nu,Z)|^2\right] = |\widehat{w}(\nu)|^2\left(1+\mathrm{E}\left[Z^2\right]\right)$.

Example 8.3: Convolution with a Random Function. As discussed earlier, random distorsion of the pulse can be modelled as the convolution of the deterministic pulse function with some random function, *i.e.*,

$$w(t,Z) = (w(\cdot) * \xi(\cdot,Z))(t) , \quad t\in\mathbb{R}.$$

Then

$$\widehat{w}(\nu,Z) = \widehat{w}(\nu)\,\widehat{\xi}(\nu,Z) , \quad \nu\in\mathbb{R}.$$

Hence $\mathrm{E}\left[\widehat{w}(\nu, Z)\right] = \widehat{w}(\nu)\,\mathrm{E}\left[\widehat{\xi}(\nu, Z)\right]$ and $\mathrm{E}\left[|\widehat{w}(\nu, Z)|^2\right] = |\widehat{w}(\nu)|^2\,\mathrm{E}\left[|\widehat{\xi}(\nu, Z)|^2\right]$.

For instance, the distortion can consists in the convolution of the pulse with a function randomly chosen among a certain set of functions, *e.g.* a set composed of different square pulses and/or triangular pulses. Then, $\mathrm{E}\left[\widehat{\xi}(\nu, Z)\right]$ and $\mathrm{E}\left[|\widehat{\xi}(\nu, Z)|^2\right]$ represent, respectively, the weighted average and the weighted square average of the Fourier transforms of the functions. To be more precise, consider that Z is a random variable that chooses, with equal probability, one out of K functions from the set $\xi_1(t), \ldots, \xi_K(t)$. Then,

$$\mathrm{E}\left[\widehat{\xi}(\nu, Z)\right] = \frac{1}{K}\sum_{k=1}^{K}\widehat{\xi}_k(\nu)\ ,$$

and

$$\mathrm{E}\left[\left|\widehat{\xi}(\nu, Z)\right|^2\right] = \frac{1}{K^2}\sum_{k=1}^{K}\left|\widehat{\xi}_k(\nu)\right|^2 .$$

Remark that the above equation shows that the functions ξ_k, $k = 1, \ldots, K$, contribute the power spectrum with their average spectral energy. Moore generally, if the functions are not chosen with equal probability, we have a weighted averaged energy.

Modulating Process

EXAMPLE 8.4: Modulating Process with Exponential Decaying Correlation. We consider a modulating w.s.s. process $\{V(t)\}_{t\in\mathbb{R}}$ with correlation function given by

$$C_V(\tau) = ae^{-\sigma^2|\tau|} .$$

Since the covariance function is integrable, the power spectral measure admits a density, *i.e.*, $\mu_V(d\nu) = S_V(\nu)\,d\nu$, where

$$S_V(\nu) = \int_{\mathbb{R}} e^{-i2\pi\tau\nu}e^{-\sigma^2|\tau|}d\tau = \frac{2}{2\pi\nu + \sigma^2} .$$

Multipath Repetitions

Multipaths introduced by the channel affect the power spectrum of its output (8.6) through the first and second orders of the random function $F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right)$. We shall now provide the expression of such moments for different type of multipath models.

EXAMPLE 8.5: Principal propagation delays. If we consider just the principal reflections, we have

$$F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right) = F\left(\nu, \bar{N}^{\mathrm{p}}\right) = 1 + \sum_{l\geq 1} e^{-i2\pi\nu T_l^{\mathrm{p}}} g^{\mathrm{p}}\left(T_l^{\mathrm{p}}, Z_l^{\mathrm{p}}\right) .$$

Hence

$$\mathrm{E}\left[F\left(\nu, \bar{N}^{\mathrm{p}}\right)\right] = 1 + \widehat{g}^{\mathrm{p}}\left(\nu, Z^{\mathrm{p}}\right) ,$$

and

$$\operatorname{Var}\left(F\left(\nu,\bar{N}^{\mathrm{p}}\right)\right)=\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{p}}\left(\nu+\upsilon\right)\right|^{2}\mu_{N^{\mathrm{p}}}\left(d\upsilon\right)+\lambda^{\mathrm{p}}\int_{\mathbb{R}}\operatorname{Var}\left(\widehat{g}^{\mathrm{p}}\left(\nu+\upsilon,Z^{\mathrm{p}}\right)\right)d\upsilon$$

For instance, when N^{p} is a Poisson point process and g^{p} a deterministic function, the above expression reads

$$\operatorname{Var}\left(F\left(\nu,\bar{N}^{\mathrm{p}}\right)\right)=\lambda^{\mathrm{p}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{p}}\left(\upsilon\right)\right|^{2}d\upsilon$$

Then, if we additionally assume w to be deterministic, we have

$$\mathrm{E}\left[\widehat{H}\left(\nu,Z^{\mathrm{c}}\right)\right]=\widehat{w}\left(\nu\right)\left(1+\widehat{g}^{\mathrm{p}}\left(\nu\right)\right)$$

and

$$\operatorname{Var}\left(\widehat{H}\left(\nu,Z^{\mathrm{c}}\right)\right)=\left|\widehat{w}\left(\nu\right)\right|^{2}\lambda^{\mathrm{p}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{p}}\left(\upsilon\right)\right|^{2}d\upsilon$$

Example 8.6: Double Poisson model. An interesting case is when both the principal and secondary delays are Poisson processes (with intensities $\lambda^{\mathrm{p}}<\lambda^{\mathrm{s}}$) and the attenuating functions are deterministic. Indeed, such a situation corresponds to the double Poisson model proposed by [Saleh and Valenzuela, 1987]. We now have

$$\mathrm{E}\left[F\left(\nu,\left(\bar{N}^{\mathrm{p}},\bar{N}^{\mathrm{s}}\right)\right)\right]=1+\lambda^{\mathrm{p}}\widehat{g}^{\mathrm{p}}\left(\nu\right)+\lambda^{\mathrm{s}}\widehat{g}^{\mathrm{s}}\left(\nu\right)+\lambda^{\mathrm{p}}\lambda^{\mathrm{s}}\widehat{g}^{\mathrm{p}}\left(\nu\right)\widehat{g}^{\mathrm{s}}\left(\nu\right)$$

and

$$\begin{aligned}\operatorname{Var}\left(F\left(\nu,\left(\bar{N}^{\mathrm{p}},\bar{N}^{\mathrm{s}}\right)\right)\right)=&\lambda^{\mathrm{p}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{p}}\left(\upsilon\right)\right|^{2}d\upsilon\lambda^{\mathrm{s}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{s}}\left(\upsilon\right)\right|^{2}d\upsilon\\&+\lambda^{\mathrm{p}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{p}}\left(\upsilon\right)\right|^{2}d\upsilon+\lambda^{\mathrm{s}}\int_{\mathbb{R}}\left|\widehat{g}^{\mathrm{s}}\left(\upsilon\right)\right|^{2}d\upsilon\end{aligned}$$

Example 8.7: Multipaths with finite renewal propagation delays and exponential gains. We consider a finite random number of principal propagation delays with deterministic gains. Therefore, the multipaths are modelled by i.i.d. truncated point processes

$$N_{n}^{\mathrm{p}}=\left\{T_{n;l}^{\mathrm{p}},l=1,\ldots,L_{n}\right\},\quad n\in\mathbb{Z},$$

where $\{L_n\}_{n\in\mathbb{Z}}$ is a sequence of i.i.d. integer-valued random variables, independent of the random times, that describes the random number of propagation delays. The gains are modelled by a deterministic function $g\left(s\right)$.

In particular, we consider an exponential attenuation function $g\left(s\right)=e^{-s\alpha}$ and i.i.d. point processes $\{N_n^{\mathrm{p}}\}_{n\in\mathbb{Z}}$ given by random pieces of renewal processes (see Example 1.1)

$$N_{n}^{\mathrm{p}}\left(ds\right)=\sum_{l=1}^{L_{n}}\epsilon_{T_{n;l}^{\mathrm{p}}}\left(ds\right),$$

where we recall that

- ϵ_a is the Dirac measure centered in a;
- $T^{\mathrm{p}}_{n;l} = S^{(n)}_1 + \ldots + S^{(n)}_l$ $(l \geq 1)$;
- $S^{(n)} \stackrel{\mathrm{def}}{=} \left\{ S^{(n)}_l \right\}_{l \geq 1}$ is i.i.d., with common c.d.f. F_S such that $F_S(\{0\}) = 0$ and common characteristic function ϕ_S;
- $\{L_n\}_{n\in\mathbb{Z}}$ is an i.i.d. sequence of integer-valued random variables, independent of $\left\{S^{(n)}\right\}_{n\in\mathbb{Z}}$, with generating function ϕ_L and such that $\mathrm{E}[L_n]$ $(= \mathrm{E}[L]) < \infty$.

We remark that when considering the overall point process, *i.e.*, the point process N convolved with the cluster points N^{p}, we obtain a renewal cluster point process (see Example 2.7)

$$N^c = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{L_n} \epsilon_{T_n + T^{\mathrm{p}}_{n;l}} .$$

In order to compute the spectrum we need to compute the first and second moments of $F(\nu, N^{\mathrm{p}})$, which is now given by

$$F\left(\nu, \left(\bar{N}^{\mathrm{p}}, \bar{N}^{\mathrm{s}}\right)\right) = F\left(\nu, \bar{N}^{\mathrm{p}}\right) = 1 + \sum_{l=1}^{L} e^{-i2\pi\nu T^{\mathrm{p}}_l} e^{\alpha T^{\mathrm{p}}_l} .$$

We have

$$\mathrm{E}\left[F\left(\nu, \bar{N}^{\mathrm{p}}\right)\right] = \frac{1}{\left(1 - \phi^*_S\left(2\pi\nu + i\alpha\right)\right)} \left(1 - \phi^*_S\left(2\pi\nu + i\alpha\right) \phi_L\left(\phi^*_S\left(2\pi\nu + i\alpha\right)\right)\right) ,$$

and

$$\begin{aligned} \mathrm{E}\left[\left|F\left(\nu, \bar{N}^{\mathrm{p}}\right)\right|^2\right] &= \mathrm{E}\left[(1+L)^2\right] + 2\mathrm{Re}\left\{ \frac{\phi_S\left(2\pi\nu + i\alpha\right)^2 \phi_L\left(\phi_S\left(2\pi\nu + i\alpha\right)\right)}{\left(1 - \phi_S\left(2\pi\nu + i\alpha\right)\right)^2} \right\} \\ &\quad - 2\mathrm{Re}\left\{ \frac{\phi_S\left(2\pi\nu + i\alpha\right)^2}{\left(1 - \phi_S\left(2\pi\nu + i\alpha\right)\right)^2} \right\} + 2\mathrm{E}[L]\,\mathrm{Re}\left\{ \frac{\phi_S\left(2\pi\nu + i\alpha\right)}{\left(1 - \phi_S\left(2\pi\nu + i\alpha\right)\right)} \right\} . \end{aligned}$$

As a more concrete example, we can consider a Poisson renewal cluster with intensity $\widetilde{\lambda}$, where the number of points are given by a Poisson random variable L with parameter θ. Now, S is exponentially distributed with mean $1/\widetilde{\lambda}$, hence

$$\phi_S\left(2\pi\nu + i\alpha\right) = \frac{1}{1 - \frac{i2\pi\nu - \alpha}{\widetilde{\lambda}}} ,$$

and

$$\begin{aligned} \phi_L\left(\phi_S\left(2\pi\nu + i\alpha\right)\right) &= \exp\left\{\theta\left(\phi_S\left(2\pi\nu + i\alpha\right) - 1\right)\right\} \\ &= \exp\left\{ -\theta \left(1 - \frac{\widetilde{\lambda}\left(\alpha + \widetilde{\lambda}\right)}{\left(\widetilde{\lambda} + \alpha\right)^2 + (2\pi\nu)^2} \right) \right\} \exp\left\{ \theta \frac{i\widetilde{\lambda}2\pi\nu}{\left(\widetilde{\lambda} + \alpha\right)^2 + (2\pi\nu)^2} \right\} . \end{aligned}$$

After simplifications we obtain

$$\begin{aligned}|\mathrm{E}[F(\nu, N^{\mathrm{p}})]|^2 =& \frac{\left(\widetilde{\lambda}+\alpha\right)^2+(2\pi\nu)^2}{\alpha^2+(2\pi\nu)^2} + \frac{\widetilde{\lambda}^2}{\alpha^2+(2\pi\nu)^2}\gamma \\ &- 2\frac{\widetilde{\lambda}\gamma}{\alpha^2+(2\pi\nu)^2}\left[\left(\widetilde{\lambda}+\alpha\right)\cos(\eta)+2\pi\nu\sin(\eta)\right] ,\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}\left[|F(\nu, N^{\mathrm{p}})|^2\right] =& 1+\theta\left(1+\frac{2\alpha}{\alpha^2+(2\pi\nu)^2}\right) \\ &- 2\left(2\left(\widetilde{\lambda}+\alpha\right)+\frac{\widetilde{\lambda}\left(\alpha+2\widetilde{\lambda}(2\pi\nu)^2\right)}{\alpha^2+(2\pi\nu)^2}\right)(\gamma\cos(\eta)-1) \\ &+ 2\left(\widetilde{\lambda}2\pi\nu+\frac{2\pi\nu\widetilde{\lambda}(2\alpha-1)}{\alpha^2+(2\pi\nu)^2}\right)\gamma\sin(\eta) ,\end{aligned}$$

where

$$\gamma = \exp\left\{-\theta\left(1-\frac{\widetilde{\lambda}\left(\widetilde{\lambda}+\alpha\right)}{\left(\widetilde{\lambda}+\alpha\right)^2+(2\pi\nu)^2}\right)\right\} , \quad \text{and} \quad \eta = \theta\frac{\widetilde{\lambda}2\pi\nu}{\left(\widetilde{\lambda}+\alpha\right)^2+(2\pi\nu)^2} .$$

Example 8.8: Multipaths with fluctuating attenuation function. We can easily take into account the situation where the attenuation function has random i.i.d. fluctuations of its amplitude. Let

$$g(s,z) = zg(s) ,$$

where z describes the random fluctuations. The i.i.d. marks now takes values over $[1-a, 1+a]$, where a is the percentage of fluctuation. For instance, if we consider a uniformly distributed fluctuation, we have

$$\widehat{\overline{g}}(\nu) = \widehat{g}(\nu)$$

and

$$\mathrm{Var}\left(\widehat{g}(\nu, Z)\right) = |\widehat{g}(\nu)|^2\,\mathrm{Var}(Z) = |\widehat{g}(\nu)|^2\,\frac{a^2}{3}$$

Obviously, several other examples can be easily produced by considering different features of the model.

Chapter 9

Random Sampling

Summary: Randomly sampled signals can be modeled as modulated random spike fields, where now the basic point process represent the sampler. Their spectrum, which was presented earlier in this manuscript, allows to compute the reconstruction error, when the reconstruction is performed by filtering the samples. Both the cases of a sampler independent and dependent on the original signal are considered.

Our contribution: Spectrum and reconstruction error for a randomly sampled signal when the signal is dependent on the sampler is a novel result. Moreover, we contribute, in both independent and dependent cases, with a unifying approach that allows to consider deterministic and random sampling under the same framework.

We say that a signal is randomly sampled when the samples are taken at random instants of time. The study of random sampling and randomly sampled signals is motivated both by practical and theoretical interests. The first ones include spectral analysis (estimation of spectra from a finite number of samples) and signal reconstruction, and the second ones include statistical analysis of reconstruction methods.

Random sampling of a continuous time random signal $X(t)$, $t \in \mathbb{R}$, yields a sequence of samples

$$X(T_n)\ , \quad n \in \mathbb{Z}\,, \tag{9.1}$$

where T_n, $n \in \mathbb{Z}$, is the sequence of points (times of events) of a point process.

At the extremities of the spectrum of randomness, we find completely random sampling, or Poisson sampling, where T_n, $n \in \mathbb{Z}$, is a homogeneous Poisson process (see Example 1.3), and the regular sampling, where $T_n = nT$, T^{-1} being the sampling frequency.

Random sampling is in most cases not deliberate. Regular sampling can become random due to jitter (lack of synchronization) or to thinning (loss of samples). Communication systems provide several examples of such situations. It can, however, be inherent to the sampling procedure, as in laser velocimetry [Gaster and Roberts, 1975], where a sample is collected only at the passage of a grain of matter through the laser beam.

In the following, the signal $X(t)$, $t \in \mathbb{R}$, is called the *original signal*, the point process T_n, $n \in \mathbb{Z}$, the *sampler*, the sequence (9.1) is the *sample sequence*, and the process

$$Y(t) = \sum_{n\in\mathbb{Z}} X(T_n)\,\delta(t - T_n)\ , \tag{9.2}$$

where $\delta(t)$ is the Dirac pulse, is called the *sampled signal* or *sample comb*.

Two theoretical question arise. The first one is related to spectral analysis, the second one to signal reconstruction.

- What is the relation between the spectrum of the sample comb (or the sample sequence) to that of the sampled signal?
- To what extent can we recover the sampled signal from the sampled comb (or the sample sequence)?

The original signal and the sampler are assumed stationary (or at least w.s.s. for the sampled process). We treat the case where the original signal and the general sampler are independent, and the case where the rate of a Poisson sampler depends on the sampled signal.

We formulate random sampling in the general spatial case. Here the sampled signal is a w.s.s. process

$$X(t) , \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m ,$$

admitting a Bochner power spectral measure μ_X.

The sampler is now a stationary point process N on $\mathbb{R}^m$, with Bartlett power spectrum μ_N. The average intensity λ of the sampler is, by definition, the average number of sampling times per unit time, and therefore it corresponds to the sampling frequency.

The sampled signal, or sample "brush" is the pseudo process

$$Y(t) = \sum_{s \in N} X(s)\, \delta(t-s) ,$$

which can be modeled as a modulated point process (see Section 2.5).

The reconstruction of the signal $X(t)$ is obtained by filtering the sample comb $\{Y(t)\}$

$$\int_{\mathbb{R}^m} \varphi(t-s)\, Y(s)\, ds ,$$

where $\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$, obtaining then a shot noise where the basic process is a modulated one (see Section 2.5).

The difference ϵ between $X(t)$ and its approximation, *i.e.* the reconstruction error, is measured by

$$\epsilon = \mathrm{E}\left[\left| \int_{\mathbb{R}^m} \varphi(t-u)\, Y(u)\, du - X(t) \right|^2 \right] .$$

9.1 Spectrum and Reconstruction Error

9.1.1 Independent case

We consider the case where the sampler is independent on the signal.

The power spectrum of the sampled signal directly follows from spectrum of a modulated point process of equation (5.3), given in Section 5.1. Concerning the reconstruction error we have the following result.

Theorem 9.1.1 *Let N be a wide-sense stationary simple point process on $\mathbb{R}^m$ with intensity $\lambda < \infty$, and Bartlett spectrum μ_N on the domain B_N. Suppose that $\{X(t)\}$ and N are independent. Then, reconstructing the signal $\{X(t)\}_{t\in\mathbb{R}}$ by filtering the sample sequence $\{Y(t)\}_{t\in\mathbb{R}}$ with a filter $\varphi \in B_N$ gives the following error*

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + |\mathrm{E}[X]|^2 \, |\lambda \widehat{\varphi}(0) - 1|^2 + \int_{\mathbb{R}^m} (1 - 2\lambda \mathrm{Re}\{\widehat{\varphi}(\nu)\})\, \mu_X(d\nu) . \tag{9.3}$$

Proof We have

$$\begin{aligned}\epsilon &= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, X(u)\, N(du) - X(t)\right|^2\right] \\ &= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u)\, X(u)\, N(du)\right|^2\right] - 2\mathrm{Re}\left\{\mathrm{E}\left[\int_{\mathbb{R}} \varphi(t-u)\, X(t)\, X(u)\, N(du)\right]\right\} + \mathrm{E}\left[|X(t)|^2\right] \\ &= A - 2\mathrm{Re}\{B\} + C\end{aligned}$$

In this expression,

$$A = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |\mathrm{E}[X]|^2 \left|\int_{\mathbb{R}^m} \varphi(t)\, dt\right|^2 = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |\mathrm{E}[X]|^2 |\widehat{\varphi}(0)|^2 ,$$

$$\begin{aligned}B = \mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t-u)\, X(t)\, X(u)\, N(du)\right] &= \lambda \int_{\mathbb{R}^m} \varphi(t-u)\, R_X(t-u)\, du \\ &= \lambda \int_{\mathbb{R}^m} \varphi(t)\, R_X(t)\, dt \\ &= \lambda \int_{\mathbb{R}^m} \varphi(t)\, C_X(t)\, dt + \lambda |\mathrm{E}[X]|^2 \int_{\mathbb{R}^m} \varphi(t)\, dt \\ &= \lambda \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\, \mu_X(d\nu) + \lambda |\mathrm{E}[X]|^2 \widehat{\varphi}(0) ,\end{aligned}$$

and

$$C = \int_{\mathbb{R}^m} \mu_X(d\nu) + |\mathrm{E}[X]|^2 .$$

Therefore

$$\begin{aligned}\epsilon = &\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) - \lambda \int_{\mathbb{R}^m} \left(\widehat{\varphi}(\nu) + \widehat{\varphi}(\nu)^*\right) \mu_X(d\nu) \\ &+ |\mathrm{E}[X]|^2 \left(1 - \lambda\left(\widehat{\varphi}(0) + \widehat{\varphi}(0)^*\right) + \lambda^2 |\widehat{\varphi}(0)|^2\right) + \int_{\mathbb{R}^m} \mu_X(d\nu) ,\end{aligned}$$

and result (9.3) follows. □

In particular, in the case $\mathrm{E}[X] = 0$ the error is

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) - 2\lambda \mathrm{Re}\left\{\int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\, \mu_X(d\nu)\right\} + \mu_X(\mathbb{R}^m) . \tag{9.4}$$

Examples of the power spectral measure and the reconstruction error can be obtained from the above results for different sampling schemes through the knowledge of μ_N. Of particular interest is the Poisson case and the effect of thinning. For ease of notation, we consider a centered signal, *i.e.* $\mathrm{E}[X] = 0$.

EXAMPLE 9.1: Poisson sampling. When the sampler is a homogeneous Poisson process, the power spectrum of the sampled signal reads

$$\mu_Y(d\nu) = \lambda^2 \mu_X(d\nu) + \lambda \mathrm{Var}(X)\, d\nu . \tag{9.5}$$

The reconstruction error is then

$$\epsilon = \int_{\mathbb{R}^m} |\lambda \widehat{\varphi}(\nu) - 1|^2 \mu_X(d\nu) + \lambda \mathrm{Var}(X) \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 (d\nu) .$$

Let us now consider the "classical" sampling framework for a centered signal $\{X(t)\}_{t\in\mathbb{R}}$ (in the univariate case), *i.e.* the signal has a spectral measure μ_X with finite support S of length $2B = l(S)$, and

$$\widehat{\varphi}(\nu) = \begin{cases} \frac{1}{\lambda} & \text{on } S\,; \\ 0 & \text{otherwise}\ . \end{cases} \tag{9.6}$$

Then, we have

$$\epsilon = \lambda \mathrm{Var}(X) \int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 \, d\nu = \lambda \mathrm{Var}(X) \int_{\mathbb{R}} \frac{1}{\lambda^2} 1_S(\nu)\, d\nu\,,$$

that is

$$\epsilon = \mathrm{Var}(X) \frac{2B}{\lambda}\,,$$

Therefore, sampling at the Nyquist rate $\lambda = 2B$ gives very poor performances, not better than the estimate based on no observation at all. This does not mean, however, that below the rate $\lambda = 2B$, there is no information (or in a sense as the result suggests "negative information") concerning the process itself contained in its samples. A better choice of a filter would indeed give a linear estimate with error less than $\sigma^2 = \mathrm{Var}(X)$. For instance, if we let $\widehat{\varphi}$ be real, we find for the error

$$\epsilon = \int_{\mathbb{R}} \left[(\lambda\widehat{\nu} - 1)^2 f_X(\nu) + \lambda^2 \sigma^2 \widehat{\varphi}(\nu)^2 \right] d\nu\,,$$

where it is assumed that $\{X(t)\}_{t\in\mathbb{R}}$ has the power spectral density $f_X(\nu)$. The minimum occurs for

$$\widehat{\varphi}(\nu) = \frac{\lambda f_X(\nu)}{\lambda^2 f_X(\nu) + \lambda\sigma^2}\,,$$

and then

$$\epsilon = \sigma^2 \left(1 - \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\, d\nu \right)\,,$$

where $\widetilde{f}_X(\nu)$ is the normalized power spectral density, *i.e.*,

$$\widetilde{f}_X(\nu) = \frac{f_X(\nu)}{\int_{\mathbb{R}} f_X(\nu')\, d\nu'} = \frac{f_X(\nu)}{\sigma^2}\,.$$

Therefore

$$\epsilon = \sigma^2 (1 - \rho)\,,$$

where

$$\rho = \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\, d\nu\,,$$

can be interpreted as the correlation coefficient between $X(t)$ and N for fixed t.

Of course, the results of classical sampling theory can be obtained as a particular case.

EXAMPLE 9.2: Classical sampling framework. When the sampling is uniform and on the line (for a centered signal, for ease of notation) from (5.3) we obtain the aliased spectrum

$$\mu_Y(d\nu) = \frac{1}{T^2} \sum_{n\in\mathbb{Z}} \mu_X \left(d\nu - \frac{n}{T} \right)\,, \tag{9.7}$$

and a reconstruction error

$$\begin{aligned}\epsilon &= \frac{1}{T^2}\int_{\mathbb{R}} \left|\widehat{\varphi}(\nu)\right|^2 \mu_X(d\nu) - \frac{2}{T}\mathrm{Re}\left(\int_{\mathbb{R}} \widehat{\varphi}(\nu)\,\mu_X(d\nu)\right) + \int_{\mathbb{R}} \mu_X(d\nu) \\ &= \int_S \left|\frac{1}{T}\widehat{\varphi}(\nu) - 1\right|^2 \mu_X(d\nu)\ .\end{aligned}$$

In the classical sampling situation, the signal $\{X(t)\}_{t\in\mathbb{R}}$ has a spectral measure μ_X with finite support S, and the filter $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ verifies (9.6), with $\lambda = 1/T$. Then the reconstruction is perfect, *i.e.*, $\epsilon = 0$. In such a case, we obtain

$$X(t) = \int_{\mathbb{R}} \varphi(t-s)\,X(s)\,N(ds) = \sum_{n\in\mathbb{Z}} X(T_n)\,\varphi(t-T_n)\ ,$$

with $\varphi(t) = sin(2\pi Bt)/(2\pi Bt)$, which is the usual reconstruction formula of Shannon [1948].

Remark that if some samples are lost then the reconstruction within the "classical" sampling settings cannot be perfect. To see this, consider the following example.

Example 9.3: Effect of thinning on the reconstruction error. Consider a random sampling scheme on the signal $X(t)_{t\in\mathbb{R}}$ characterized by a sampler N with sampling rate λ and spectral measure μ_X and a reconstruction error ϵ.

We now suppose that the sampler is affected by random thinning, *i.e.*, samples are randomly lost. Call $\widetilde{\epsilon}$ the reconstruction error in such a case. Then

$$\begin{aligned}\widetilde{\epsilon} = q^2\epsilon + \left(1-q^2\right)\mathrm{Var}(X) + \lambda pq\mathrm{Var}(X)\int_{\mathbb{R}} \left|\widehat{\varphi}(\nu)\right|^2 d\nu \\ + \lambda q(1-q)\int_{\mathbb{R}^m} 2\mathrm{Re}\left\{\widehat{\varphi}(\nu)\right\}\mu_X(d\nu)\ . \qquad (9.8)\end{aligned}$$

Remark that when the probability of a loss is zero, *i.e.* $q = 1$, we have $\epsilon = \widetilde{\epsilon}$, while when all the samples are lost, *i.e.* $q = 0$, we have $\epsilon = \mathrm{Var}(X)$, which is indeed the error of the estimate based on no observation at all.

An interesting situation is when we are within the classical sampling framework with perfect reconstruction. In such a case, μ_N is the power spectrum of a regular grid, given by equation (3.13), and $\epsilon = 0$. The above formula then gives the reconstruction error due to random losses of samples.

9.1.2 Dependent Case

We now turn our attention to the case where the sampler is dependent on the signal to be sampled. In particular, we consider that the sampling is performed by a Poisson process with rate depending on the original signals. The model for the sampler is now a Cox process on $\mathbb{R}^m$ (see for instance Example 1.4 or [Daley and Vere-Jones, 1988, 2002]) with the conditional (w.r.t. X) intensity of the form

$$\lambda(t) = \lambda(t, X)\ .$$

For instance, in the univariate case $\lambda(t) = \left|\dot{X}(t)\right|^2$ where $\dot{X}$ is the derivative at t of $t \to X(t)$, hence obtaining a sampling rate that depends on the variability of the signal to be sampled. More complicated functionals can be considered.

The power spectrum of the sampled signal is then given by expression (5.6).

Theorem 9.1.2 *Consider the setup of Theorem 5.1.2 (extended Bochner spectrum for spike fields with dependent modulation). Then, reconstructing the signal by filtering the sample sequence* $\{Y(t)\}_{t\in\mathbb{R}}$ *with a filter* $\varphi \in \mathrm{L}^1_{\mathbb{C}}(\mathbb{R}^m) \cap \mathrm{L}^2_{\mathbb{C}}(\mathbb{R}^m)$ *gives the following error*

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu) + \mathrm{E}\left[X(t)^2 \lambda(t)\right] \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 d\nu$$
$$+ \int_{\mathbb{R}^m} 2\mathrm{Re}\{\widehat{\varphi}(\nu)\} \mu_{XZ}(d\nu) + \mathrm{Var}(X) + |\mathrm{E}[X] - \mathrm{E}[Z]\widehat{\varphi}^*(0)|^2 ,$$

where $Z(t) = X(t)\lambda(t)$, *and* μ_{XZ} *is the cross spectrum of* $(X)_{t\in\mathbb{R}}$ *and* $(Z)_{t\in\mathbb{R}}$

Proof The proof follows the same line as the proof of error in the independent case (Theorem 9.1.1. □

In the particular case of $X(t) \equiv 1$, we recover the formula for the Bartlett spectrum of a Cox process (3.16), while when $\lambda(t) = X(t)$ we have

$$\mu_Y(d\nu) = \mu_{X^2}(d\nu) + \mathrm{E}\left[X^3\right] d\nu .$$

9.2 Sampling Scheme for Channel Estimation

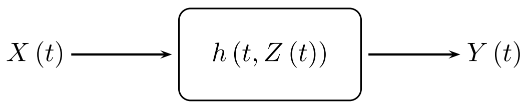
In this section we present an approach based on Poisson sampling to the second order estimation of the transmittance of a fading, time-invariant channel.

Many communication systems are *bandwidth-expanding*, in the sense that the transmitted signal occupies a bandwidth larger than the *symbol rate*. Retrieving estimation data within the classical Shannon-Nyquist sampling scheme requires sampling rates that are very high and expensive to implement.

The Poisson sampling approach is motivated by its alias-free property. By the latter we mean that the spectrum of the sampled and the original signals are in a one-to-one relation, *independently* of the sampling rate. Such a features then plays a key role in estimating second order properties of *bandwidth-expanding* communication systems, such as the UWB ones.

Shot noise model of the channel

We consider an UWB transmission over a fading, time-invariant, channel. The channel is characterized by its random impulse response function $H(t)$, which can be equivalently noted as $h(t, Z(t))$, where h is a deterministic function depending on a family of i.i.d. random parameters $\{Z(t)\}_{t\in\mathbb{R}}$ that models the randomness of the channel.



When the input signal X is a stream of spikes (or a signal related to a stream of spikes) the output Y of the channel is a shot noise with random excitation (see Definition 2.2) and its power spectrum can be obtained from equation (4.8)

$$\mu_Y(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu, Z)\right]\right|^2 \mu_X(d\nu) + \lambda \mathrm{Var}\left(\widehat{h}(\nu, Z)\right) d\nu , \tag{9.9}$$

where μ_X can be the spectrum of one on the UWB signal presented in Chapter 7, and $\widehat{h}(\nu, Z)$ is the transmittance of the channel, *i.e.*, the Fourier transform of its impulse response function h.

Estimation of the variance through Poisson sampling

We aim to estimate the variance of the random transmittance $\text{Var}\left(\widehat{h}(\nu, Z)\right)$ of the channel from a set of samples taken at the output of the channel (samples of the received signal Y). If we suppose that the spectrum μ_X of the transmitted signal is known as well as the first order moment of the random transmittance, then from equation (9.9) we can see that the second order moment can be straightforwardly computed from the spectra μ_Y of the received signal .

In an UWB framework, regular sampling of the received signal requires a sampling rate that is very high: a lower rate than the Nyquist one signifies an aliased spectrum and a consequent estimation error. Poisson sampling can provide an interesting alternative since the corresponding spectrum is alias-free (see equation (9.5)).

Let T be the average period of the stream of pulses at the input of the channel, and let λ be the average Poisson sampling rate. Call $\widetilde{Y}$ the signal obtained by a Poisson sampling of the received signal Y. For ease of notation, we assume that the power spectral measure of the input and output signals admit a density, *i.e.*, $\mu(d\nu) = S(\nu)\,d\nu$. Then, combining equation (9.5) and (9.9) we obtain

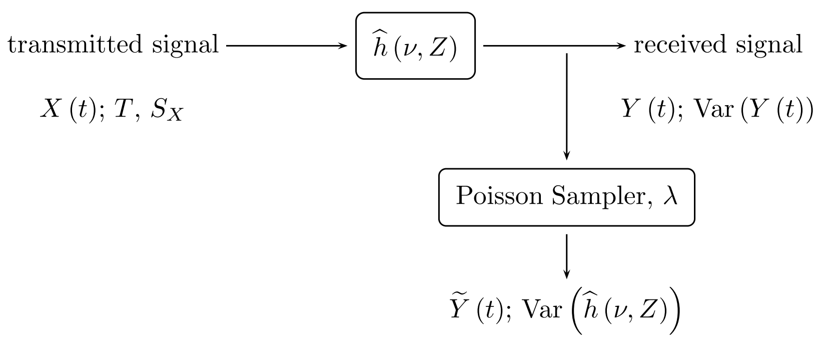
$$S_{\widetilde{Y}}(\nu) = \lambda^2 \left(\left| \mathrm{E}\left[\widehat{h}(\nu, Z)\right] \right|^2 S_X(\nu) + \frac{1}{T} \text{Var}\left(\widehat{h}(\nu, Z)\right) \right) + \lambda \text{Var}\left(Y(t)\right) ,$$

and finally

$$\text{Var}\left(\widehat{h}(\nu, Z)\right) = \frac{T}{\lambda^2} \left(S_{\widetilde{Y}}(\nu) - \lambda^2 \left| \mathrm{E}\left[\widehat{h}(\nu, Z)\right] \right|^2 S_X(\nu) - \lambda \text{Var}\left(Y(t)\right) \right) ,$$

where $\text{Var}\left(Y(t)\right)$ is the power of the transmitted signal which can be estimated from the samples.

The following block-diagram summarizes the estimation approach



It is then necessary to develop an appropriate spectral estimator from Poisson samples. We remark that several results on the spectral analysis of Poisson randomly sampled signals are available (the first ones being obtained by Masry [1978b,a]). Practical implementation the requires testing and choosing the most appropriate estimator for the problem presented here.

Conclusions

In Summary

We have presented a systematic study of second order properties of a large class of complex signals and random fields connected with point processes and of interest to communications, biology, seismology, and wavelet spectral analysis, among other domains of applications.

In particular, complex signals have been described as the result of various operations on a basic spike field. In practice, such operations correspond to adding, in a modular way, specific features to a basic model. As demonstrated with some applications in communications, this systematic approach greatly simplifies the computations and allows to treat highly complex models.

We have then derived the power spectra, sometimes in a generalized sense, of such a large class of signals. First, we have recalled the notion of Bartlett spectrum of point processes, providing details on the domain of definition that are missing in the literature. Secondly, we have presented a general formula, the fundamental isometry formula, which is the "seed" formula for the computation of the spectra of random spike fields and related processes. Such formula has the following remarkable properties:

- it allows to derive, in a systematic manner, the spectra of several complex signals from a single expression;
- it unifies results scattered in the literature, allowing for various extensions;
- it provides a tractable tool for the computation of spectra of highly complex signals that were not yet available, such as the spectra of pulse train through multipath fading channels;
- it preserves the modularity of the construction of the complex signal in the sense that each additional feature added to a basic model appears as a separate and explicit contribution to the corresponding basic spectrum.

In summary, we have provided a tractable approach for modeling highly complex signals related to random spike fields and a "toolbox" for easily obtaining their power spectra, with features that have a tremendous impact to model design and analysis.

Theoretical Extensions

Theoretical research in progress concerns other classes of signals related to point process, in particular semi-Markov point processes, and space–time point process models of interest to mobile communications. It is also of great interest to be able to compute the spectra of processes related to discrete time point processes. Palm calculus is a fundamental tool for such a development.

Applications to biology

As already discussed, spiky signals occur is several domains. Biology is among the most appealing ones. In particular, second order analysis of neural spike trains can be used to characterize the data with respect to the stimuli used to generate it. Our spectra "toolbox" can then provide an important support for such a characterization.

Software Toolbox

The collection of exact spectrum expressions we have derived can be exploited to develop a "software toolbox"[I]. Concerning the large family of UWB signals, such software can allow to visualize and study the spectrum of different combinations of pulse modulations and multi-user signaling, including additional random effects that affect the transmission. Extensions can include the effect of the transmission over a multipath fading channel.

[I]The software, `PulseSpectra`, is now available at `lcavwww.epfl.ch/software/`. It can be freely used under the GNU-GPL licence.

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