Extending a continuous constraint solver to handle mixed global optimization

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$$\mathcal{P} \equiv \begin{cases} \min & f(x) \\ \text{s.c.} & c_i(x), i = 1..m \\ & \underline{\mathbf{b}} \leq x \leq \overline{\mathbf{b}} \\ & x_i \in \mathbb{Z}, i = 1..p \\ & x_i \in \mathbb{R}, i = p+1..n \end{cases} \quad (1)$$

where $f : \mathbb{Z}^p \times \mathbb{R}^{n-p} \to \mathbb{R}$ is a nonlinear function, and $c_i(x) \equiv_{def} g_i(x) \leq 0$ where $g_i : \mathbb{Z}^p \times \mathbb{R}^{n-p} \to \mathbb{R}$.

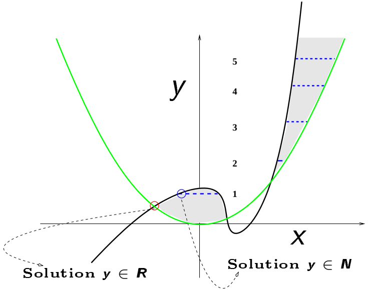
- Find a rigorous enclosure $[L, U]$ of the objective function values at optimum
- Find a feasible solution x^* at U
- ... with tolerance ϵ

Extending the ICOS solver: constraint satisfaction, continuous optimization, mixed optimization ?

minimize x

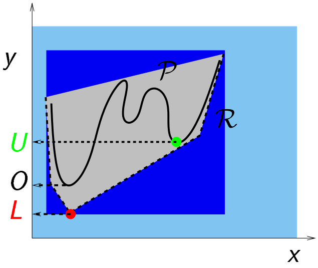
subject to $y - x^2 \geq 0; \quad -y + x^2(x-2) - 2 \leq 0$

$x, y \in [-10, +10]; \quad y \in \mathbb{N}$



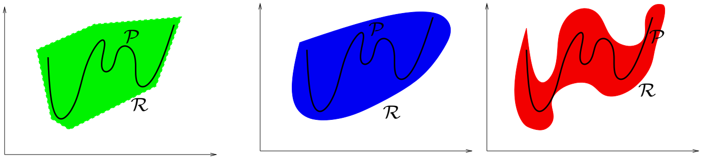
$y \in \mathbb{N}$	x ∈ [	-0.61803400241354988775 6,	-0.61803398241354978726 0]
	y ∈ [	+1.00000000000000000000,	+1.00000000000000000000]
$y \in \mathbb{R}$	x ∈ [	-0.73206337323128989336 1,	-0.73204337323128998438 2]
	y ∈ [	+0.53588772396888251226 0,	+0.53590772396888242123 9]

- While $\mathcal{L} \neq \emptyset$ do %$\mathcal{L}$ initialized with the input box
  - Select a box B from the set of current boxes $\mathcal{L}$
  - Reduction (filtering or tightening) of B
  - Lower bounding and Upper bounding of f in the box B
  - Splitting of B if it is not proved empty; put the new boxes in $\mathcal{L}$



$$\mathcal{R} \equiv \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) = 0,\ \ i \in \mathcal{RE} \\ & g_i(x) \geq 0,\ \ i \in \mathcal{RI} \\ & \underline{x_i} \leq x_i \leq \overline{x_i}, x_i \in \mathcal{RX} \end{cases}$$

The objective value at the global minimum of $\mathcal{R}$ is lower than the objective value of the global minimum of $\mathcal{P}$.



Linear relaxation

Embed

$$\mathcal{P} \equiv \begin{cases} \text{minimize} & h(x) \\ \text{subject to} & c_i(x) = 0, \;\; i \in \mathcal{E} \\ & c_i(x) \geq 0, \;\; i \in \mathcal{I} \\ & \underline{x_i} \leq x_i \leq \overline{x_i}, x_i \in \mathcal{X} \end{cases}$$

into a linear relaxation

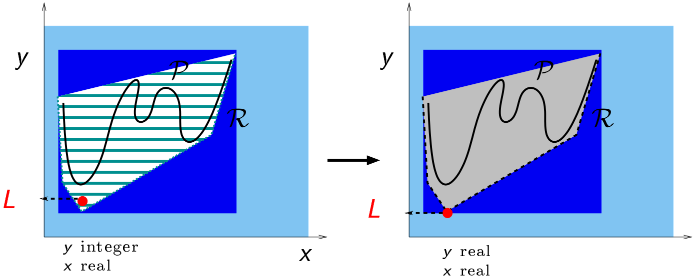
$$\mathcal{LR} \equiv \begin{cases} \text{minimize} & cx \\ \text{subject to} & A_E x = b_E, \\ & A_I x \geq b_I, \\ & \underline{x_i} \leq x_i \leq \overline{x_i}, x_i \in \mathcal{RX} \end{cases}$$

Solving $\mathcal{LR}$ provides a rigorous lower bound of the objective function value at the optimum of $\mathcal{P}$, provided that $\mathcal{LR}$ is rigorously generated and solved. The precision of the lower bound depends on the tightness of $\mathcal{LR}$.

The main steps

- Unchanged steps:
  - Box selection: select the box with the lowest lower bound
  - Branching: splitting the box
- Adapted steps:
  - Lower bounding: relaxing the integrity constraint ... solving a linear relaxation of the problem
  - Reduction: relaxing the integrity constraint ... reduction ... rounding off
  - Upper bounding: local search ... rounding off ... feasibility proof

1. Relaxing the integrity constraint by forcing the integer variables to take real values
2. By exploiting the linear relaxation, get a rigorous lower bound



Continuous optimization ...

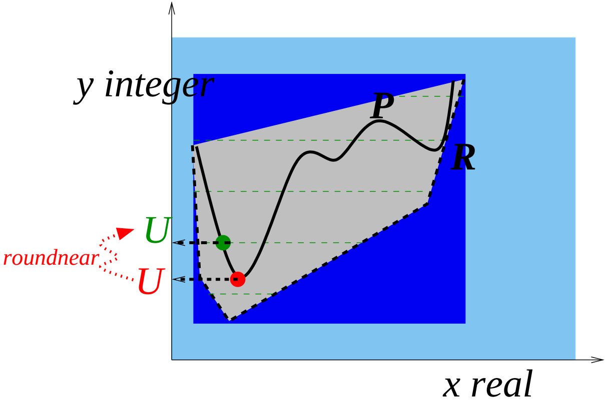
1. Call a continuous local search method to find an approximate feasible point P_{loc}, possibly nearly optimal (COIN/IpOpt)

$$\mathcal{P} \equiv \begin{cases} \text{minimize} & h(x) \\ \text{subject to} & c_i(x) = 0, \;\; i \in \mathcal{E} \\ & c_i(x) \geq 0, \;\; i \in \mathcal{I} \\ & x \in \mathcal{D} \\ & x^0 := P_{lb} \quad \text{\%where } P_{lb} \text{ is the underestimator} \end{cases}$$

2. Prove the feasibility of a box $\mathcal{D}'$ containing P_{loc} by inflation with a feasibility procedure (e.g., Krawczyck, Miranda, Borsuk, ...)

... Mixed optimization

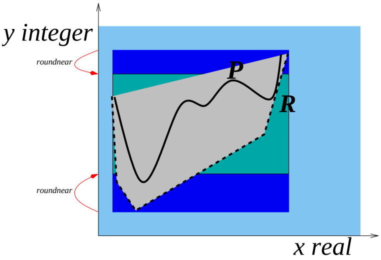
- Relaxing the integrity constraint on the integer variables
- Call the step (1) of the usual upper-bounding step and find an approximate solution by calling a local search
- Roundding the real values of the integer variables to the nearest ones
- Call the feasibility step (2) of the continuous optimization by inflation of the initially continuous variables


y integer
P
R
U
roundnear
U
x real

Two main classes:

- Feasibility based reduction
  $\Longrightarrow$ discards subdomains where the constraints are not satisfied
  $\Longrightarrow$ optimization, constraint satisfaction (solving equations)
- Optimality based reduction
  $\Longrightarrow$ discards subdomains where the objective function takes values above the current upper bound
  $\Longrightarrow$ optimization

1. Relaxing the integrity constraint of the integer variables
2. Apply the classical continuous reduction algorithms (e.g., 2B, Interval Newton, Quad, ...)
3. Fix the relaxed integer variables to the nearest integers



Property 1: Partial correction

When the algorithm terminates, it provides a correct solution.

A direct consequence of the correction of the upper-bounding and the lower-bounding steps.

- n: num of vars; ni: num of int vars; m: num of constraints;
- BN: Best known bound
- $[L, U]$: rigorous bounds found by ICOS
- Time out: 60 s $\epsilon = 10^{-3}$

Problem	n	ni	m	BN	$[L, U]$	$T(s)$
prob10	3	1	3	3.45	[3.443e+00, 3.446e+00]	0
nvs10	3	2	3	-310.8	[-3.128e+02, -3.108e+02]	60
nvs16	3	2	1	0.7	[1.897e-01, 7.031e-01]	60
nvs03	3	2	3	16	[1.414e+01, 1.600e+01]	60
nvs06	3	2	1	1.77	[1.746e+00, 1.770e+00]	60
nvs04	3	2	1	0.72	[5.119e-02, 2.120e+00]	60
nvs21	4	2	3	-5.68	[-5.685e+00, -5.679e+00]	60
ex1222	4	1	4	1.08	[8.105e-01, 1.077e+00]	60
nvs15	4	3	2	1	[2.144e-01, 1.000e+00]	60
nvs07	4	3	3	4	[3.500e+00, 4.000e+00]	60
nvs01	4	2	4	12.47	[1.311e+01, inf]	60
nvs11	4	3	4	-431	[-4.314e+02, -4.310e+02]	0
nvs08	4	2	4	23.45	[2.297e+01, 2.367e+01]	60
st_testph4	4	3	11	-80.5	[-8.287e+01, -8.050e+01]	60

Problem	n	ni	m	BN	$[L, U]$	$T(s)$
gear	5	4	1	0	[-0.000e+00, 7.843e-05]	0
gbd	5	3	5	2.2	[2.200e+00, 2.200e+00]	0
nvs12	5	4	5	-481.2	[-4.820e+02, -4.812e+02]	60
ex1226	6	3	6	-17	[-1.700e+01, -1.700e+01]	0
st_test1	6	5	2	0	[-1.943e+01, inf]	60
ex1221	6	3	6	7.67	[7.931e+00, inf]	60
nvs13	6	5	6	-585.2	[-5.858e+02, -5.852e+02]	3
st_miqp4	7	3	5	-4574	[-4.574e+03, -4.571e+03]	0
nvs18	7	6	7	-778.4	[-7.927e+02, -7.780e+02]	60
synthes1	7	3	7	6.01	[5.677e+00, 6.016e+00]	60
st_test4	7	6	6	-7	[-7.179e+00, -7.000e+00]	60
st_test2	7	6	3	-9.25	[-9.273e+00, -9.250e+00]	60
gear4	7	4	2	1.64	[-9.881e-324, inf]	60
nvs17	8	7	8	-1100.4	[-4.264e+03, -6.732e+02]	60
ex1223a	8	4	10	4.58	[4.580e+00, 4.580e+00]	0
ex1223b	8	4	10	4.58	[4.580e+00, 4.580e+00]	0
nvs02	9	5	4	5.98	[5.961e+00, 5.986e+00]	60
alan	9	4	8	2.93	[2.925e+00, 2.983e+00]	60
gear3	9	4	5	0	[-0.000e+00, inf]	60
nvs19	9	8	9	-1098.4	[-2.717e+04, inf]	60

Problem	n	ni	m	BN	$[L, U]$	$T(s)$
nvs22	9	4	10	6.06	[8.514e+00, inf]	60
ex1225	9	6	11	31	[3.078e+01, 3.400e+01]	60
nvs14	9	5	4	-40153.72	[-4.036e+04, -3.597e+04]	60
nvs23	10	9	10	-1125.2	[-3.189e+05, inf]	60
nvs24	11	10	11	-1033.2	[-5.611e+05, inf]	60
st_test6	11	10	6	471	[3.825e+02, 5.670e+02]	60
st_test5	11	10	12	-110	[-1.785e+02, -1.100e+02]	60
st_testgr1	11	10	6	-12.81	[-1.942e+01, -1.225e+01]	60
nvs09	11	10	1	-43.13	[-4.316e+01, -4.313e+01]	45
gkocis	12	3	9	-1.92	[-3.565e+00, 2.002e+00]	60
ex1223	12	4	14	4.58	[4.580e+00, 4.580e+00]	0
synthes2	12	5	15	73.04	[4.910e+01, inf]	60
ex1224	12	8	8	-0.94	[-9.437e-01, -9.428e-01]	19
st_test3	14	13	11	-7	[-7.004e+00, -7.000e+00]	1
windfac	15	3	14	0.25	[1.025e-02, inf]	60
nvs20	17	5	9	230.92	[1.867e+02, 4.203e+02]	60
synthes3	18	8	24	68.01	[1.532e+01, inf]	60
st_testgr3	21	20	21	-20.59	[-2.074e+01, -2.059e+01]	60
fac1	23	6	19	1.61E+008	[1.716e+07, inf]	60
st_test8	25	24	21	-29605	[-2.963e+04, -2.960e+04]	5

Problem	n	ni	m	BN	$[L, U]$	$T(s)$
ex1263a	25	24	36	19.6	[1.920e+01, 3.010e+01]	60
ex1264a	25	24	36	8.6	[8.103e+00, 1.210e+01]	60
gear2	29	24	5	0	[-0.000e+00, 1.873e-06]	43
csched1a	29	15	23	-30430.18	[-1.349e+05, inf]	60
ex1265a	36	35	45	10.3	[1.030e+01, 1.160e+01]	60
meanvarx	36	14	45	14.37	[1.437e+01, inf]	60
ex1252	40	15	44	128893.74	[-0.000e+00, inf]	60
batch	47	24	74	285506.51	[1.326e+05, inf]	60
ex1266a	49	48	54	16.3	[1.612e+01, inf]	60
tloss	49	48	54	16.3	[1.612e+01, inf]	60
nous1	51	2	44	1.57	[6.227e-02, inf]	60
nous2	51	2	44	0.63	[6.018e-01, inf]	60
ex1233	53	12	65	155010.67	[3.832e+04, inf]	60
fac2	67	12	34	3.32E+008	[2.202e+06, inf]	60
fac3	67	12	34	3.20E+007	[1.697e+06, 1.378e+08]	60
waterx	71	14	55	910.47	[1.454e+02, inf]	60
minlphix	85	20	93	316.69	[-inf, inf]	60
gasnet	91	10	70	6999381.56	[7.431e+05, inf]	60
ex1244	96	23	130	82042.91	[4.950e+04, inf]	60
feedtray	98	7	92	-13.41	[-7.500e+01, inf]	60

Problem	n	ni	m	BN	$[L, U]$	$T(s)$
gastrans	107	21	150	89.09	[8.909e+01, inf]	60
m7_ar5_1	113	42	270	106.46	[-1.310e-06, inf]	60
m7_ar4_1	113	42	270	106.76	[-1.288e-06, inf]	60
m7_ar3_1	113	42	270	143.59	[-1.021e-06, inf]	60
m7_ar25_1	113	42	270	143.59	[-1.132e-06, inf]	60
no7_ar2_1	113	42	270	107.82	[-6.661e-07, inf]	60
m7_ar2_1	113	42	270	190.24	[-1.199e-06, inf]	60
no7_ar4_1	113	42	270	98.52	[-2.442e-07, inf]	60
no7_ar3_1	113	42	270	107.82	[-2.331e-07, inf]	60
no7_ar25_1	113	42	270	107.82	[-4.108e-07, inf]	60
no7_ar5_1	113	42	270	90.62	[-3.331e-07, inf]	60
fo7_ar4_1	113	42	270	20.73	[-5.551e-09, inf]	60
o7_ar2_1	113	42	270	24.84	[-8.216e-07, inf]	60
fo7_ar3_1	113	42	270	22.52	[-3.331e-08, inf]	60
fo7_ar25_1	113	42	270	23.09	[-1.110e-08, inf]	60
fo7_ar2_1	113	42	270	24.84	[-2.776e-08, inf]	60
fo7_ar5_1	113	42	270	17.75	[-7.075e-16, inf]	60
eniplac	142	24	190	-132117.08	[-1.774e+05, inf]	60
4stufen	150	48	99	116329.67	[-inf, inf]	60
deb10	183	22	130	209.43	[-1.087e-320, inf]	60

- 15 problems $\longrightarrow$ correct bounds
- 12 problems $\longrightarrow$ tight bounds
- 22 problems $\longrightarrow$ lower bound or upper bound enough tight
- remaining problems $\longrightarrow$ bounds far from the solution

$\Longrightarrow$ This simple extension of the branch and bound algorithm seems interesting.

Conclusion

1. An interesting simple and cheap extension of a continuous solver to handle mixed optimization
2. The experimental results are encouraging

Perspectives

- Exploit local search procedures coming from discrete domains
- Enhancing the lower bounding ... discrete relaxations ...
- Enhancing the upper bounding ... :-(
- Exploit reduction techniques from discrete domains