Output bounds for uncertain dynamical systems

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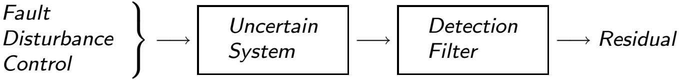
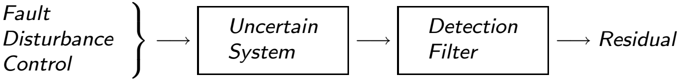
Context: Fault detection



- The residual is almost zero in the non faulty case.
- Threshold crossing is used to detect fault occurrence.
- The disturbance level determines the threshold level.

Detection filter design by Kalman filtering: Residual $=$ innovation.
$\Longrightarrow$ residual independent of the control when no uncertainty.
Not true when uncertainties exist.

Objective: Define an adaptive threshold for the residual testing, taking into account for the residual dependency w.r.t. input in the presence of system uncertainties.



Fault
Disturbance
Control
Uncertain System
Detection Filter
Residual

Objective: Define an adaptive threshold for the residual testing, taking into account for the residual dependency w.r.t. input in the presence of system uncertainties.

Mean: The adaptive thresholds are the instantaneous upper and lower bounds of the residual w.r.t. to the uncertainties, knowing the control input.

Outline of the talk:

I Bounds for a simple gain with high frequency uncertainty.

II Bounds for a 1st order low pass filter with uncertain settling time.

III Bounds propagation over cascade of uncertain systems.

A simple gain with high frequency uncertainty

$$s = \left[G + \delta \frac{T\,p}{1 + T\,p} \right] e$$

At low frequency the gain is G. At high frequency the gain is $G + \delta$, with $\underline{\delta} \leq \delta \leq \overline{\delta}$.

Two ways for computing upper ($\overline{s}$) and lower ($\underline{s}$) bounds of the output s, knowing the input e:

- Use of impulse response bounds: $\longrightarrow$ nonlinear filtering
- Use of an LFR model: $\longrightarrow$ hybrid automata

Impulse response bounds

The impulse response is

$$s = h * e = \int_{-\infty}^{+\infty} h(t-\tau,\delta)\, e(\tau)\, d\tau$$

with

$$h(\tau,\delta) = (G+\delta)\,\delta(\tau) - \frac{\delta}{T}\, e^{-\frac{\tau}{T}}\; Y(\tau)$$

It is a sum of signed terms easy to bound. We have

$$\underline{h}(\tau) \le h(\tau,\delta) \le \overline{h}(\tau)$$

with

$$\begin{array}{lcl} \underline{h}(\tau) = (G+\underline{\delta})\,\delta(\tau) - \frac{\overline{\delta}}{T}\, e^{-\frac{\tau}{T}}\; Y(\tau) & \Longrightarrow & \underline{H}(p) \\ \overline{h}(\tau) = (G+\overline{\delta})\,\delta(\tau) - \frac{\underline{\delta}}{T}\, e^{-\frac{\tau}{T}}\; Y(\tau) & \Longrightarrow & \overline{H}(p) \end{array}$$

Use of impulse response bounds for output bounding

Decompose the input as follow

$$e(t) = e^{+}(t) + e^{-}(t) = \frac{e(t) + |e(t)|}{2} + \frac{e(t) - |e(t)|}{2}$$

Use the fact that $e^{+} \geq 0$ and $e^{-} \leq 0$, to show that

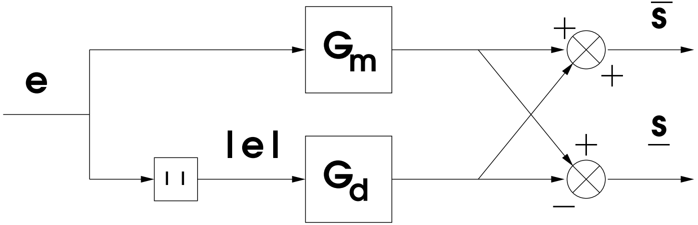
$$\underbrace{s_m - s_d}_{\underline{s}} \leq s \leq \underbrace{s_m + s_d}_{\overline{s}}$$

where

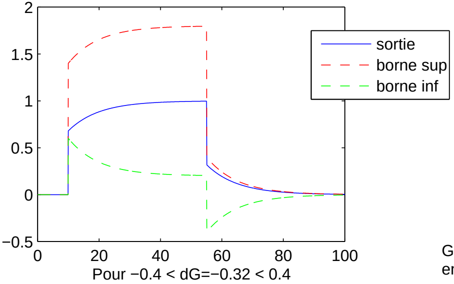
$$\begin{array}{llll} s_m = H_m\, e & \text{with} & H_m = \frac{\overline{H}+\underline{H}}{2} & \text{medium transfert} \\ s_d = H_d\, |e| & \text{with} & H_d = \frac{\overline{H}-\underline{H}}{2} & \text{difference transfert} \end{array}$$

Remarks:
* The bounds generator is nonlinear ($|e|$)
* The methodology applies whenever impulse response bounds are known.

Architecture of the output bounds generator

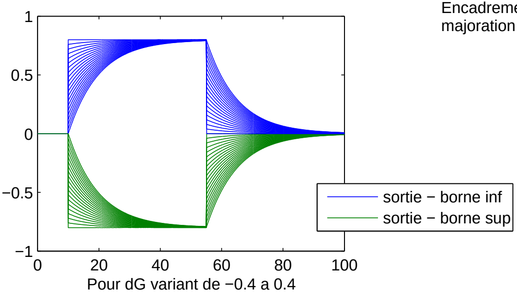


Application to the uncertain gain



Gain incertain
en haute fréquence

Encadrement de la sortie par
majoration de la RI



A bad result, because the static gain is not kept equal to G for the bounds

Impulse bound approach revisited

Decompose the system response in order to isolate the uncertain part:

$$s = \left[G + \delta \frac{T\,p}{1 + T\,p} \right] e = G\,e + \delta\,e^{'}$$

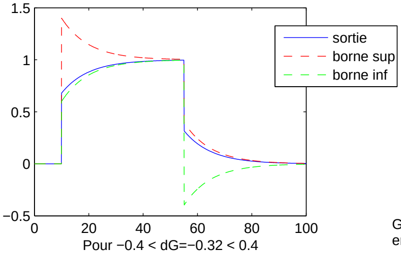
$e^{'}$ is the pseudo-derivation of e.

Consider the uncertain part $s^{'} = \delta\,e^{'}$. Its impulse response $(h^{'}(\tau) = \delta\,\delta(\tau))$ is easy to bound.

Applying the previous methodology in order to build the bounds generator, yields to $s^{'}_{m} = 0$ and $s^{'}_{d} = |e^{'}|$ (with normalized uncertainty). And thus

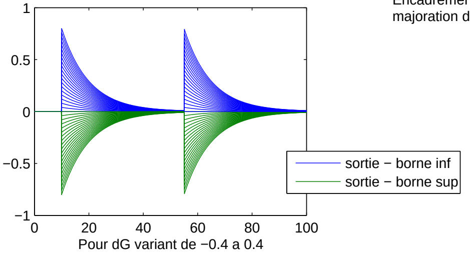
$$\begin{array}{rcl} \overline{s^{'}} & = & G\,e + |e^{'}| \\ \underline{s^{'}} & = & G\,e - |e^{'}| \end{array}$$

Simulation results



Gain incertain
en haute fréquence

Encadrement de la sortie par
majoration de la RI avec pseudo−derivee



OK!

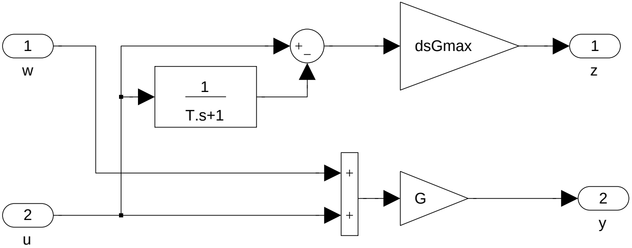
The state space approach

A linear fractional representation of the system (with normalized uncertainty) is

$$\left\{ \begin{array}{rcl} \dot{x} & = & -\frac{1}{T}\,(x-e) \\ z & = & \delta_{max}\,(e-x) \\ s & = & G\,(e+w) \end{array} \right.$$

If $w = \delta\, z$ with δ varying over $[-1\,,\,1]$, then the high frequency gain varies from $(1-\delta_{max})\,G$ up to $(1+\delta_{max})\,G$.

The simulation model is



Output bounding with an hybrid automata

Search for a bounds genrator with the same architecture as the system.

$$\begin{array}{rcl}\underline{s} & = & G\,e + \delta_{min}\,e^{'} \\ \overline{s} & = & G\,e + \delta_{max}\,e^{'}\end{array}$$

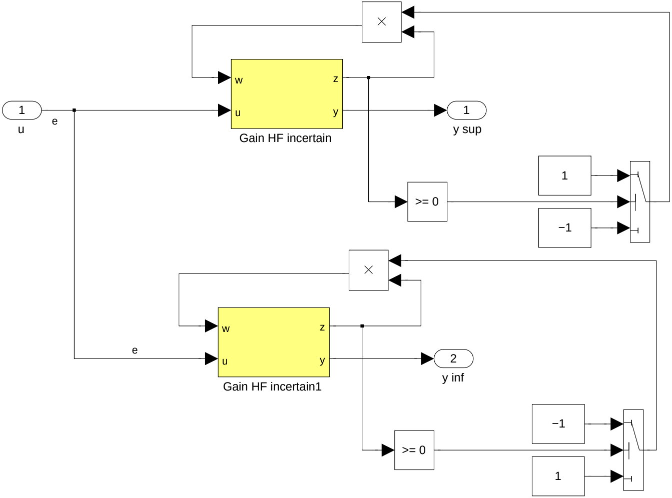
with varying δ_{min} and δ_{max} to be tuned so that $\underline{s}(t) \leq s(t) \leq \overline{s}(t)$.

The good choice is obviously :

$$\delta_{min} = \left\{ \begin{array}{ll} \underline{\delta} & \text{if } e^{'} \geq 0 \\ \overline{\delta} & \text{if not} \end{array} \right. \qquad \delta_{max} = \left\{ \begin{array}{ll} \overline{\delta} & \text{if } e^{'} \geq 0 \\ \underline{\delta} & \text{if not} \end{array} \right.$$

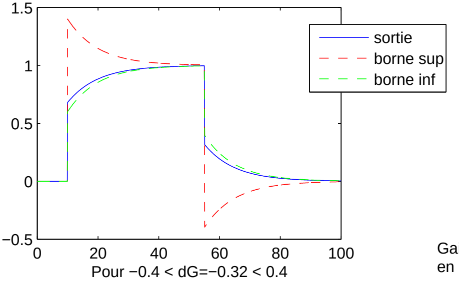
The switching rule needs the sign of $e^{'}$ to be computed.

Hybrid automata architecture



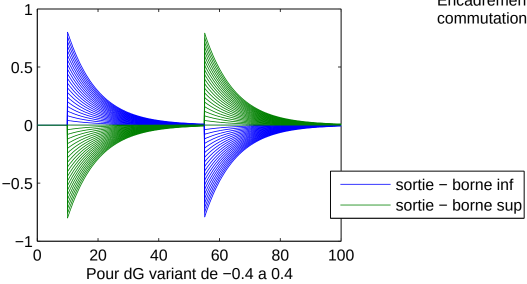
The switching rule uses the output z which has the same sign as e'.

Hybrid automata results



Gain incertain
en haute fréquence

Encadrement de la sortie par
commutation de paramètres



Same (good!) results has previously.

Outline of the talk:

I Bounds for a simple gain with high frequency uncertainty

II Bounds for a 1st order low pass filter with uncertain settling time.
  - Use of impulse response bounds: nonlinear filtering
  - Use of an LFR model: hybrid automata

III Bounds propagation over cascade of uncertain systems

1st order system with uncertain settling time

The transfer function is

$$s = \frac{1}{1 + T\,p}\,e$$

where T is uncertain, $T \in [\underline{T}\,,\ \overline{T}]$.

The impulse response is:

$$h(t) = \frac{1}{T}\,e^{-\frac{t}{T}}\,Y(t)$$

Direct bounding of each (positive) terms of h gives:

$$\underline{h}(t) = \frac{1}{\overline{T}}\,e^{-\frac{t}{\underline{T}}}\,Y(t) \quad \text{and} \quad \overline{h}(t) = \ldots$$

These impulse response bounds are bad because, once again, the static gain is not preserved.
Simple and *good* impulse response bounds do not exist.

Bounding via augmented dynamic

We introduce a pseudo-derivation

$$H(p) = \frac{1}{1 + T\,p} = 1 - \frac{p}{p + \frac{1}{T}} = 1 - \underbrace{\frac{p + \frac{1}{\tau}}{p + \frac{1}{T}}}_{H^{'}(p)} \quad \underbrace{\frac{p}{p + \frac{1}{\tau}}}_{pseudo\,-\,derivation}$$

We have $s = e - H^{'}\,e^{'}$, where $H^{'}(p)$ is uncertain. Its impulse response is:

$$h^{'}(t) = \delta(t) + \left(\frac{1}{\tau} - \frac{1}{T}\right)\,e^{-\frac{t}{T}}\,Y(t)$$

Choose $\tau \leq \underline{T}$, then all terms are positive, and thus easy to bound.

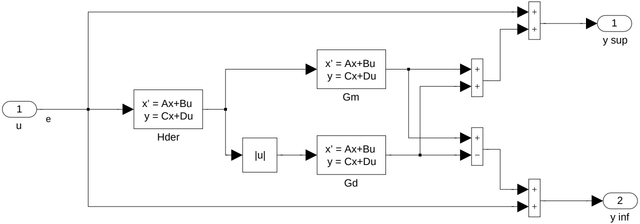
$$\underline{h}^{'}(t) = \delta(t) + \left(\frac{1}{\tau} - \frac{1}{\overline{T}}\right)\,e^{-\frac{t}{\overline{T}}}\,Y(t) \quad \text{and} \quad \overline{h}^{'}(t) = \ldots$$

The static gain is preserved because of the pseudo-derivation.

Architecture of the bounds generator

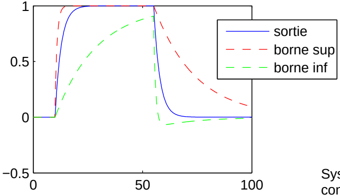
From the impulse response bounds, using the nonlinear bound generator design procedure, we deduce output bounds for $s^{'} = H^{'} e^{'}$.

Since $s = e - s^{'}$, bounds for s are $\overline{s}(t) = e(t) - \underline{s}^{'}(t)$ and $\underline{s}(t) = e(t) - \overline{s}^{'}(t)$.



Simulation results

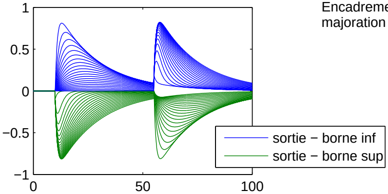
Because of the pseudo-derivation there is no static error.



Pour −9 < dT=−7.2 < 9

Systeme du 1er ordre avec
constante de temps incertaine

Encadrement de la sortie par
majoration de la RI pseudo−derivee



Pour dT variant de −9 a 9

The state space approach

A linear fractional representation writes

$$\left\{ \begin{array}{lcl} \dot{x} & = & -\frac{1}{T}\,(x - e - w) \\ z & = & \delta_{max}\,(x - e - w) \\ s & = & x \end{array} \right.$$

With $w = \delta\, z$ and δ varying over $[-1\,,\,1]$, the settling time varies from $(1 - \delta_{max})\,T$ up to $(1 + \delta_{max})\,T$.

Hybrid automata

We search for a bounds generator of the following structure

$$\begin{array}{rcl} \dot{\underline{s}} & = & \dfrac{1}{T_{min}}\,(e-\underline{s}) \\ \dot{\overline{s}} & = & \dfrac{1}{T_{max}}\,(e-\overline{s}) \end{array}$$

where T_{min} and T_{max} must be tuned so that $\underline{s} \leq s \leq \overline{s}$.

It can be shown that

$$s-\underline{s} = h * (T_{min}-T)\,\dot{\underline{s}}$$

So, *thanks to the positivity of h*

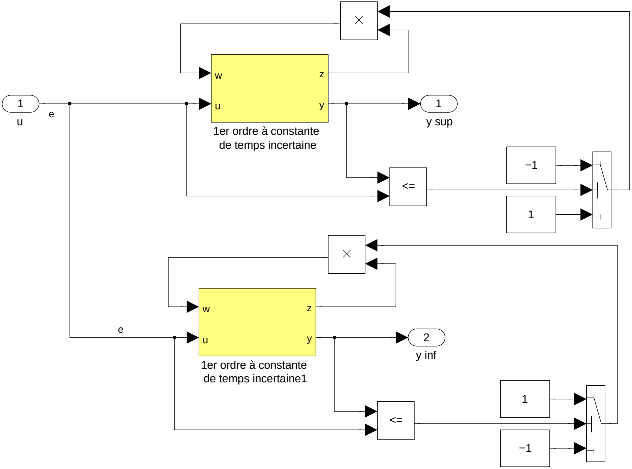
$$\left.\begin{array}{l} (T_{min}-T)\,\dot{\underline{s}} \geq 0 \\ h \geq 0 \end{array}\right\} \Longrightarrow s-\underline{s} \geq 0$$

Since $\dot{\underline{s}}$ and $e-\underline{s}$ have the same sign, the good switching rule for T_{min} is

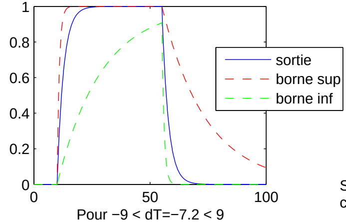
$$T_{min} = \left| \begin{array}{ll} \overline{T} & \text{if } e-\underline{s} \geq 0 \\ \underline{T} & \text{if not} \end{array}\right.$$

Implementation

Use the uncertain simulation box driven by the switched value of the settling time.

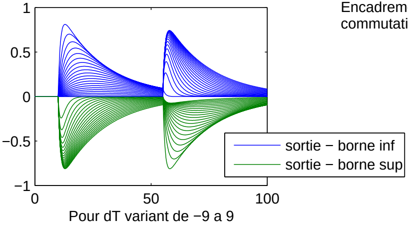


Simulation results



Systeme du 1er ordre avec constante de temps incertaine

Encadrement de la sortie par commutation de paramètres



Link with Kieffer, Ramdani, Meslem, Walter, ... results

Kieffer, Ramdani, Meslem, Walter, ... give rules to build an hybrid automata based on a theorem from Muller.

The switching condition seems to be $s - e \geq 0$ instead of $\overline{s} - e \geq 0$.

But we want not to use the measure of s (fault detection context).

Misunderstanding?

Extension to *n*'s order systems

Consider the uncertain transfer function

$$s = \frac{N(p)}{D(p)}\, e$$

where

$$D(p) = \sum a_k\, p^k \quad \text{with uncertain } a_k \in [\underline{a}_k\,,\, \overline{a}_k]$$

One can show that

$$\left.\begin{array}{l} \text{Impulse response positivity for } 1/D \\ (a_k^{min} - a_k)\, \overline{s}^{(k)} \geq 0 \end{array}\right\} \Longrightarrow s - \underline{s} \geq 0$$

where

$$\underline{s} = \frac{N(p)}{D_{min}(p)}\, e \quad \text{and } D_{min}(p) = \sum a_k^{min}\, p^k$$

Then the switching rule must be

$$a_k^{min} = \left|\begin{array}{ll} \overline{a}_k & \text{If } \overline{s}^{(k)} \geq 0 \\ \underline{a}_k & \text{If not} \end{array}\right.$$

Needs $\overline{s}^{(k)}$ to be computed from e.

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I Bounds for a simple gain with high frequency uncertainty

II Bounds for a 1st order lowpass filter with uncertain time response.

III Bounds propagation over cascade of uncertain systems

- One uncertain system
- Two uncertain systems

Bounds propagation through out a system

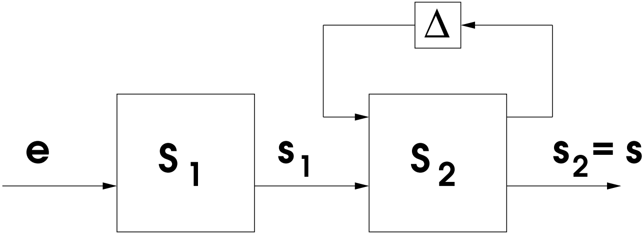
Very easy if:
- the system has no uncertainty
- its impulse response h is signed (≥ 0 or ≤ 0)

For example, if $h(t) \geq 0 \; \forall t$, then

$$\underline{e} \leq e \leq \overline{e} \quad \Longrightarrow \quad h * \underline{e} \leq s \leq h * \overline{e}$$

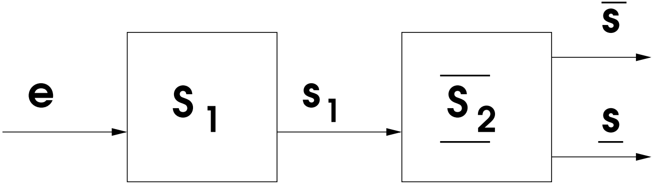
A cascade with (only) one uncertain system

The input *e* of the systems cascade is known.



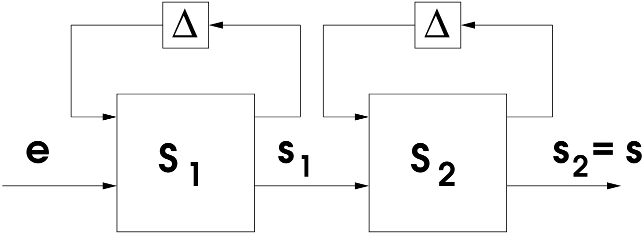
A cascade with (only) one uncertain system

A bound generator $\overline{\underline{S}}_2$ of S_2 is introduced.
Because s_1, the input of S_2, is computed, there is no bounds propagation...



If S_1 is the uncertain system, use a permutation (linear framework) and apply the same methodology.

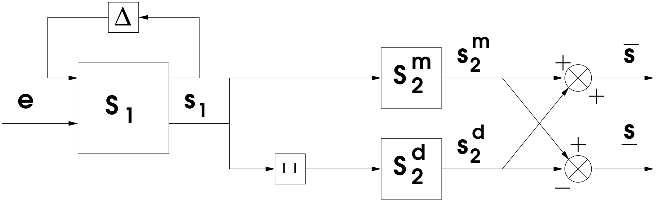
A cascade of two uncertain system



Assume that $\underline{h}_2 \leq h_2 \leq \overline{h}_2$, so that the bound generator of S_2 has a nonlinear architecture.

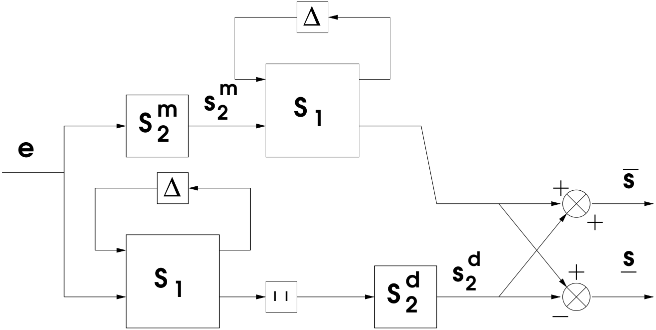
Introduce the nonlinear bound generator of S_2.

A cascade of two uncertain system



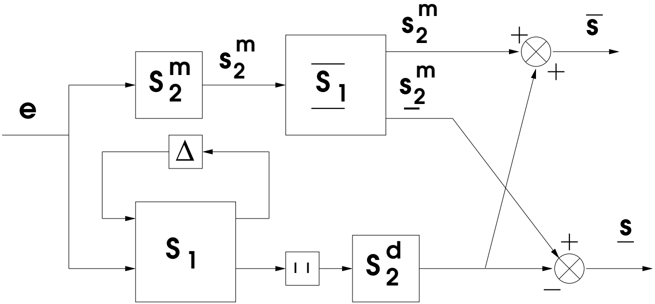
Duplicate S_1 on the upper and lower tracks.
Permute S_1 and S_2^m.

A cascade of two uncertain system



Introduce $\overline{\underline{S}}_1$ the bounds generator of S_1.

A cascade of two uncertain system



Since $s_2^m - s_2^d \le s \le s_2^m - s_2^d$, we have:

$$\underline{s}_2^m - s_2^d \le s \le \overline{s}_2^m - s_2^d$$

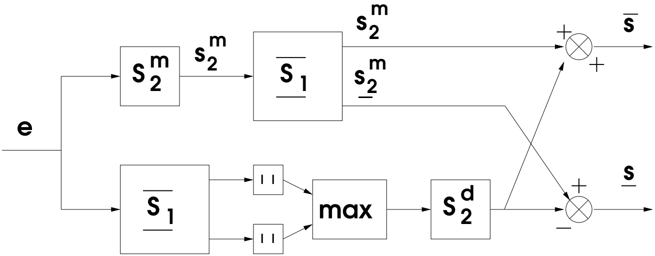
Futhermore

$$\overline{s}_2^d = \overline{|s_1|} = \max\{|\underline{s}_1|\,,\ |\overline{s}_1|\}$$

So that

$$\underline{s}_2^m - \overline{s}_2^d \le s \le \overline{s}_2^m - \overline{s}_2^d$$

And the final architecture is



Conclusion and future work

- Design of output bounds generator for uncertain linear systems:
  - Impulse response approach ⟶ nonlinear generator
  - State space approach ⟶ hybrid automata
- Some tricks
  - Impulse response positivity
  - Dynamic augmentation
- Questions
  - Link between the two approaches and other published results
  - Analysis of conservatism
  - Extension to general LFR systems
  - Impact of input noise, initial state error
  - Application to fault detection problem