Bounds on Eigenvalues of Symmetric Interval Matrices

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Introduction

Applications

- Mechanics and engineering:
  - robotics;
  - automobile suspension systems;
  - mass structures;
  - vibrating systems.
- principal component analysis;
- global optimization.

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Definition

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. It has n real eigenvalues

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Introduction

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The midpoint and the radius of $\mathbf{A}$

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

A symmetric interval matrix

$$\mathbf{A}^S := \{A \in \mathbf{A} \mid A = A^T\}.$$

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Aim

For each i determine bounds for the set of the i-th eigenvalues

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) := \left\{\lambda_i(A) \mid A \in \mathbf{A}^S\right\}.$$

Introduction

History

- Deif, 1991: exact bound under restrictive assumptions;
- Hertz, 1992: exponential formula for $\overline{\lambda}_1(\mathbf{A}^S)$ and $\underline{\lambda}_n(\mathbf{A}^S)$;
- Qiu et al., 1996: approximation;
- Rohn, 2005: simple formulae for outer estimation;
- Kolev, 2006: outer estimation for general case with non-linear dependencies;
- Leng & He, 2007: outer estimation;
- Yuan et al., 2008: inner estimation;
- Hladík & Daney & Tsigaridas, 2009: outer estimation.

Initial bounds

Theorem (Rohn, 2005)

We have

$$\boldsymbol{\lambda}_i(\mathbf{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)].$$

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Properties

- Easy and cheap to compute;
- good starting point;
- all intervals the same width.

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Proposition

We have

$$\overline{\lambda}_1(\mathbf{A}^S) \leq \lambda_1(|\mathbf{A}|).$$

Diagonal maximization

Proposition

Let $i \in \{1, \dots, n\}$. Then there is some matrix $A \in \mathbf{A}^S$ with diagonal entries $A_{j,j} = \overline{A}_{j,j}$ such that $\lambda_i(A) = \overline{\lambda}_i(\mathbf{A}^S)$.

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Consequence

For each $i \in \{1, \dots, n\}$,

$$\overline{\lambda}_i(\mathbf{A}^S) = \overline{\lambda}_i(\mathbf{A}^S_r),$$

where

$$\mathbf{A}^S_r := \{A \in \mathbf{A}^S \mid A_{j,j} = \overline{A}_{j,j} \ \forall j = 1, \dots, n\}.$$

Interlacing approach

Theorem (Interlacing property, Cauchy, 1829)

Let $A \in \mathbb{R}^n$ be a symmetric matrix and let A_i be a matrix obtained from A by removing the i-th row and column. Then

$$\lambda_1(A) \geq \lambda_1(A_i) \geq \lambda_2(A) \geq \lambda_2(A_i) \geq \cdots \geq \lambda_{n-1}(A_i) \geq \lambda_n(A).$$

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Generalization to symmetric interval matrices

$$\overline{\lambda}_1(\mathbf{A}^S) \geq \overline{\lambda}_1(\mathbf{A}_i^S) \geq \overline{\lambda}_2(\mathbf{A}^S) \geq \overline{\lambda}_2(\mathbf{A}_i^S) \geq \cdots \geq \overline{\lambda}_{n-1}(\mathbf{A}_i^S) \geq \overline{\lambda}_n(\mathbf{A}^S),$$

and

$$\underline{\lambda}_1(\mathbf{A}^S) \geq \underline{\lambda}_1(\mathbf{A}_i^S) \geq \underline{\lambda}_2(\mathbf{A}^S) \geq \underline{\lambda}_2(\mathbf{A}_i^S) \geq \cdots \geq \underline{\lambda}_{n-1}(\mathbf{A}_i^S) \geq \underline{\lambda}_n(\mathbf{A}^S).$$

Interlacing approach, direct version

Algorithm (Interlacing approach, direct version for upper bounds)

1. $\mathbf{B}^S := \mathbf{A}^S$;
2. for $k = 1, \ldots, n$ do
   1. compute upper bound $\lambda_1^u(\mathbf{B}^S)$;
   2. $\lambda_k^u(\mathbf{A}^S) := \lambda_1^u(\mathbf{B}^S)$;
   3. select the most promising index $i \in \{1, \ldots, n-k+1\}$;
   4. remove the i-th row and the i-th column from $\mathbf{B}^S$;
3. put $I := \emptyset$; for $k = 1, \ldots, n$ do
   1. select the most promising index $i \in \{1, \ldots, n\} \setminus I$, and put $I := I \cup \{i\}$;
   2. create $\mathbf{B}^S$ from $\mathbf{A}^S$ by restriction on index set I;
   3. compute $\lambda_1^u(\mathbf{B}^S)$;
   4. $\lambda_{n-k+1}^u(\mathbf{A}^S) := \min\left\{\lambda_{n-k+1}^u(\mathbf{A}^S),\ \lambda_1^u(\mathbf{B}^S)\right\}$;
4. return upper bounds $\lambda_k^u(\mathbf{A}^S),\ k = 1, \ldots, n$.

Interlacing approach, direct version

Selection of the most promising index i

Two basic possibilities:

- More computations, but sharper bounds

$$i := \arg \min_{j=1,\dots,n-k+1} \lambda_1^u(\mathbf{B}_j^S),$$

Interlacing approach, direct version

Selection of the most promising index i

Two basic possibilities:

- More computations, but sharper bounds

$$i := \arg \min_{j=1,\dots,n-k+1} \lambda_1^u(\mathbf{B}_j^S),$$

- Less computations, but less sharper bounds

$$i := \arg \min_{j=1,\dots,n-k+1} \sum_{r,s\neq j} |\mathbf{B}_{r,s}|^2.$$

Interlacing approach, indirect version

Theorem (Weyl, 1912)

Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. Then

$$\lambda_{r+s-1}(A+B) \le \lambda_r(A) + \lambda_s(B) \quad \forall r,s \in \{1,\dots,n\},\ r+s \le n+1,$$
$$\lambda_{r+s-n}(A+B) \ge \lambda_r(A) + \lambda_s(B) \quad \forall r,s \in \{1,\dots,n\},\ r+s \ge n+1.$$

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Generalization to symmetric interval matrices

Considering $A := A_c$ and $B \in [-A_\Delta, A_\Delta]^S$ we get

$$\overline{\lambda}_{r+s-1}(\mathbf{A}^S) \le \lambda_r(A_c) + \overline{\lambda}_s([-A_\Delta, A_\Delta]^S) \quad \forall r,s:\ r+s \le n+1,$$
$$\underline{\lambda}_{r+s-n}(\mathbf{A}^S) \ge \lambda_r(A_c) + \underline{\lambda}_s([-A_\Delta, A_\Delta]^S) \quad \forall r,s:\ r+s \ge n+1.$$

Interlacing approach, indirect version

Algorithm (Interlacing approach, indirect version)

1. Compute eigenvalues $\lambda_1(A_c) \geq \cdots \geq \lambda_n(A_c)$;
2. compute bounds $\lambda_1^u\big([-A_\Delta, A_\Delta]^S\big), \ldots, \lambda_n^u\big([-A_\Delta, A_\Delta]^S\big)$;
3. for $k = 1, \ldots, n$ do
   1. $\lambda_k^u(\mathbf{A}^S) := \min_{i=1,\ldots,k} \left\{\lambda_i(A_c) + \lambda_{k-i+1}^u\big([-A_\Delta, A_\Delta]^S\big)\right\};$
4. return upper bounds $\lambda_k^u(\mathbf{A}^S),\ k = 1, \ldots, n$.

Filtering

Theorem

Let $\lambda^0 \notin \boldsymbol{\lambda}_i(\mathbf{A}^S)$, and define $\mathbf{M}^S := \mathbf{A}^S - \lambda^0 I$. Then $(\lambda^0 + \lambda) \notin \boldsymbol{\lambda}_i(\mathbf{A}^S)$ for all real λ satisfying

$$|\lambda| < \frac{1 - \frac{1}{2}\,\rho\left(|I - QM_c| + |I - QM_c|^T + |Q|M_\Delta + M_\Delta|Q|^T\right)}{\frac{1}{2}\,\rho\left(|Q| + |Q|^T\right)},$$

where $Q \in \mathbb{R}^{n\times n}$, $Q \neq 0$, is an arbitrary matrix.

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Remark

The best choice is $Q := M_c^{-1}$ or its approximation.

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Proof.

Based od the Beeck–Rump sufficient condition on regularity of an interval matrix $\mathbf{B}$: $\rho(|I - QB_c| + |Q|B_\Delta) < 1$. $\square$

Filtering

Algorithm (Filtering $\boldsymbol{\lambda}_i(\mathbf{A}^S)$ from above)

1. Compute an initial approximation $\boldsymbol{\lambda} \supseteq \boldsymbol{\lambda}_i(\mathbf{A}^S)$;
2. $t := 0$;
3. $\mu := \varepsilon\lambda_\Delta + 1$ ($\varepsilon > 0$ is given);
4. while $\mu > \varepsilon\lambda_\Delta$ and $t < T$ do
    1. $t := t + 1$;
    2. $\mathbf{M}^S := \mathbf{A}^S - \overline{\lambda} I$;
    3. compute $Q := M_c^{-1}$;
    4. $\mu := \dfrac{2 - \rho\left(|I - QM_c| + |I - QM_c|^T + |Q|M_\Delta + M_\Delta|Q|^T\right)}{\rho\left(|Q| + |Q|^T\right)}$;
    5. if $\mu > 0$ then $\overline{\lambda} := \overline{\lambda} - \mu$;
    6. if $\overline{\lambda} < \underline{\lambda}$ then return $\boldsymbol{\lambda} := \emptyset$;
5. return $\boldsymbol{\lambda}$.

Numerical experiments

Example

$$\mathbf{A}^S = \begin{pmatrix} [0,2] & [-7,3] & [-2,2] \\ [-7,3] & [4,8] & [-3,5] \\ [-2,2] & [-3,5] & [1,5] \end{pmatrix}^S.$$

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	$[\lambda_1^l(\mathbf{A}^S),\ \lambda_1^u(\mathbf{A}^S)]$	$[\lambda_2^l(\mathbf{A}^S),\ \lambda_2^u(\mathbf{A}^S)]$	$[\lambda_3^l(\mathbf{A}^S),\ \lambda_3^u(\mathbf{A}^S)]$
Rohn	$[-2.2298,\ 16.0881]$	$[-6.3445,\ 11.9734]$	$[-8.9026,\ 9.4154]$
Direct	$[4.0000,\ 15.3275]$	$[-2.5616,\ 6.0000]$	$[-8.9026,\ 2.0000]$
Indirect	$[-0.7436,\ 16.0881]$	$[-3.3052,\ 10.4907]$	$[-8.9026,\ 6.3760]$
Best initial	$[4.0000,\ 15.3275]$	$[-2.0000,\ 6.0000]$	$[-8.3759,\ 2.0000]$
Filtering	$[4.0000,\ 15.3275]$	$[-2.0000,\ 6.0000]$	$[-7.9186,\ 2.0000]$
Optimal	$[?,\ 15.3275]$	$[?,\ ?]$	$[-7.8184,\ ?]$
Inner	$[5.6056,\ 15.3275]$	$[0.8301,\ 6.0000]$	$[-7.8184,\ 1.0000]$

Numerical experiments

Example (Qiu et al., 1996)

$$\mathbf{A}^S = \begin{pmatrix} [2975, 3025] & [-2015, -1985] & 0 & 0 \\ [-2015, -1985] & [4965, 5035] & [-3020, -2980] & 0 \\ 0 & [-3020, -2980] & [6955, 7045] & [-4025, -3975] \\ 0 & 0 & [-4025, -3975] & [8945, 9055] \end{pmatrix}^S$$

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	$[\lambda_1^l(\mathbf{A}^S),\ \lambda_1^u(\mathbf{A}^S)]$	$[\lambda_2^l(\mathbf{A}^S),\ \lambda_2^u(\mathbf{A}^S)]$	$[\lambda_3^l(\mathbf{A}^S),\ \lambda_3^u(\mathbf{A}^S)]$	$[\lambda_4^l(\mathbf{A}^S),\ \lambda_4^u(\mathbf{A}^S)]$
R	[12560.630, 12720.433]	[6984.557, 7144.361]	[3309.947, 3469.750]	[825.260, 985.063]
D	[8945.000, 12720.227]	[4945.000, 9055.000]	[2924.505, 6281.722]	[825.260, 3025.000]
I	[12560.630, 12720.433]	[6984.557, 7144.361]	[3309.947, 3469.750]	[825.260, 985.063]
B	[12560.630, 12720.227]	[6990.762, 7138.180]	[3320.286, 3459.432]	[837.064, 973.199]
F	[12560.813, 12720.227]	[6999.786, 7129.272]	[3332.716, 3447.463]	[841.533, 968.585]
O	[12560.838, 12720.227]	[7002.283, 7126.828]	[3337.078, 3443.313]	[842.925, 967.108]

R . . . Rohn, D . . . direct, I . . . indirect, B . . . best initial, F . . . filtering, 0 . . . optimal

Singular values

Definition

Let $\mathbf{A} \subset \mathbb{R}^{m \times n}$, its singular value sets are

$$\boldsymbol{\sigma}_i(\mathbf{A}) := \{\sigma_i(A) \mid A \in \mathbf{A}\}, \quad i = 1, \ldots, q := \min\{m, n\}.$$

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Jordan–Wielandt matrix

Singular values of $A \in \mathbb{R}^{m \times n}$ are identical with the q largest eigenvalues of

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$$\begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix}^S.$$