# Inner and outer approximation of the polar diagram of a sailboat


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# Outline

- Problem Statement
- Example: Polar Diagram of a Sailboat
- Quantified Set Inversion Algorithm
- Sailboat Control Application

# Problem Statement

- Many control theory problems have a solution set that can be written into the form

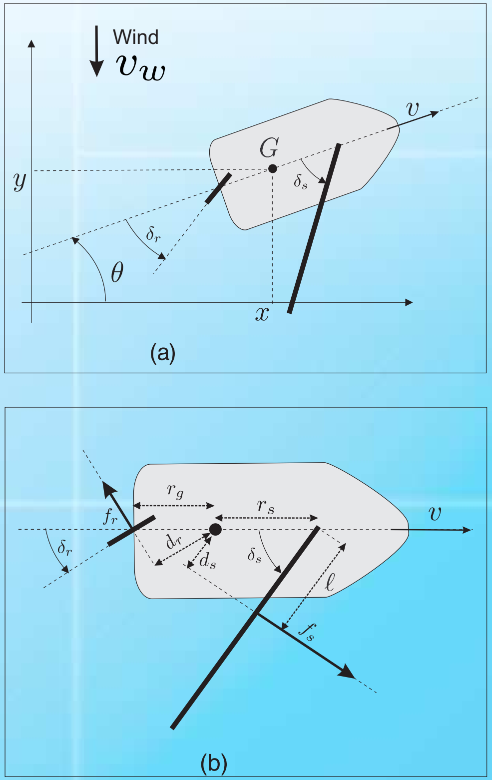
$$\mathbb{S} \triangleq \{\boldsymbol{p} \in \boldsymbol{P} \mid (\exists \boldsymbol{q} \in \boldsymbol{Q})\mathbf{f}(\boldsymbol{p}, \boldsymbol{q}) = \mathbf{0}\},$$

- where $\mathbf{P}$ is a box of $\mathbb{R}^{n_p}$, $\mathbf{Q}$ is a box of $\mathbb{R}^{n_q}$ and $\mathbf{f}$ is a continuous function from $\mathbb{R}^{n_p+n_q}$ to $\mathbb{R}^{n_f}$.

- Challenging problem: General algorithm able to find an **inner** and an outer approximation of $\mathbb{S}$.

# Example: Sailboat Modelization

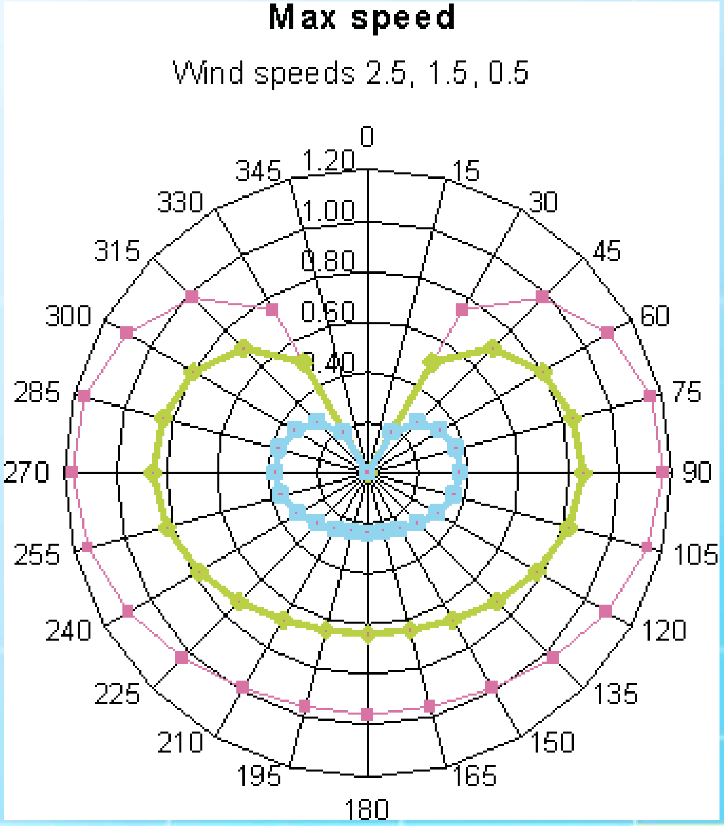
$$\left\{\begin{array}{lcc}
\dot{x} & = & v\cos\theta, \\
\dot{y} & = & v\sin\theta - \beta v_w, \\
\dot{\theta} & = & \omega, \\
\dot{\delta}_s & = & u_1, \\
\dot{\delta}_r & = & u_2, \\
\dot{v} & = & \frac{f_s \sin\delta_s - f_r \sin\delta_r - \alpha_f v}{m}, \\
\dot{\omega} & = & \frac{(\ell - r_s\cos\delta_s) f_s - r_r \cos\delta_r f_r - \alpha_\theta \omega}{J}, \\
f_s & = & \alpha_s\left(v_w\cos(\theta+\delta_s) - v\sin\delta_s\right), \\
f_r & = & \alpha_r v \sin\delta_r.
\end{array}\right.$$




- The state vector $\mathbf{x} = (x, y, \theta, \delta_s, \delta_r, v, \omega)^{\mathsf{T}} \in \mathbb{R}^7$. The inputs $u_1$ and $u_2$ of the system are the derivatives of the angles $\delta_s$ and $\delta_r$.

# Example: Polar Diagram of a Sailboat

- The *polar diagram* of the sailboat is the set $\mathbb{S}$ of all pairs $(\theta, v)$ that can be reached by the boat, in a cruising mode.



# Example: Polar Diagram of a Sailboat

- In boat's cruising mode, the speed of the boat, its angular velocity, . . . are assumed to be constant, i.e.,

$$\dot{\theta} = 0, \dot{\delta}_s = 0, \dot{\delta}_r = 0, \dot{v} = 0, \dot{\omega} = 0.$$

Then, we get

$$\begin{cases} 0 & = & \frac{f_s \sin \delta_s - f_r \sin \delta_r - \alpha_f v}{m}, \\ 0 & = & \frac{(\ell - r_s \cos \delta_s) f_s - r_r \cos \delta_r f_r}{J}, \\ f_s & = & \alpha_s \left( v_w \cos \left( \theta + \delta_s \right) - v \sin \delta_s \right), \\ f_r & = & \alpha_r v \sin \delta_r. \end{cases}$$

# Example: Polar Diagram of a Sailboat

- The polar diagram can be written as

$$\mathbb{S} = \{(v,\theta) \mid \exists \delta_r, \exists \delta_s, \mathbf{f}(v,\theta,\delta_r,\delta_s) = \mathbf{0}\},$$

where

$$\mathbf{f}(v,\theta,\delta_r,\delta_s) = \begin{pmatrix} \alpha_s (v_w \cos(\theta+\delta_s) - v \sin\delta_s) \sin\delta_s - \alpha_r v \sin^2 \delta_r - \alpha_f v \\ (\ell - r_s \cos\delta_s)\, \alpha_s (v_w \cos(\theta+\delta_s) - v\sin\delta_s) - r_r \alpha_r v \sin\delta_r \cos\delta_r \end{pmatrix}$$

# Example: Polar Diagram of a Sailboat

- An elimination of $\delta_r$ yields to

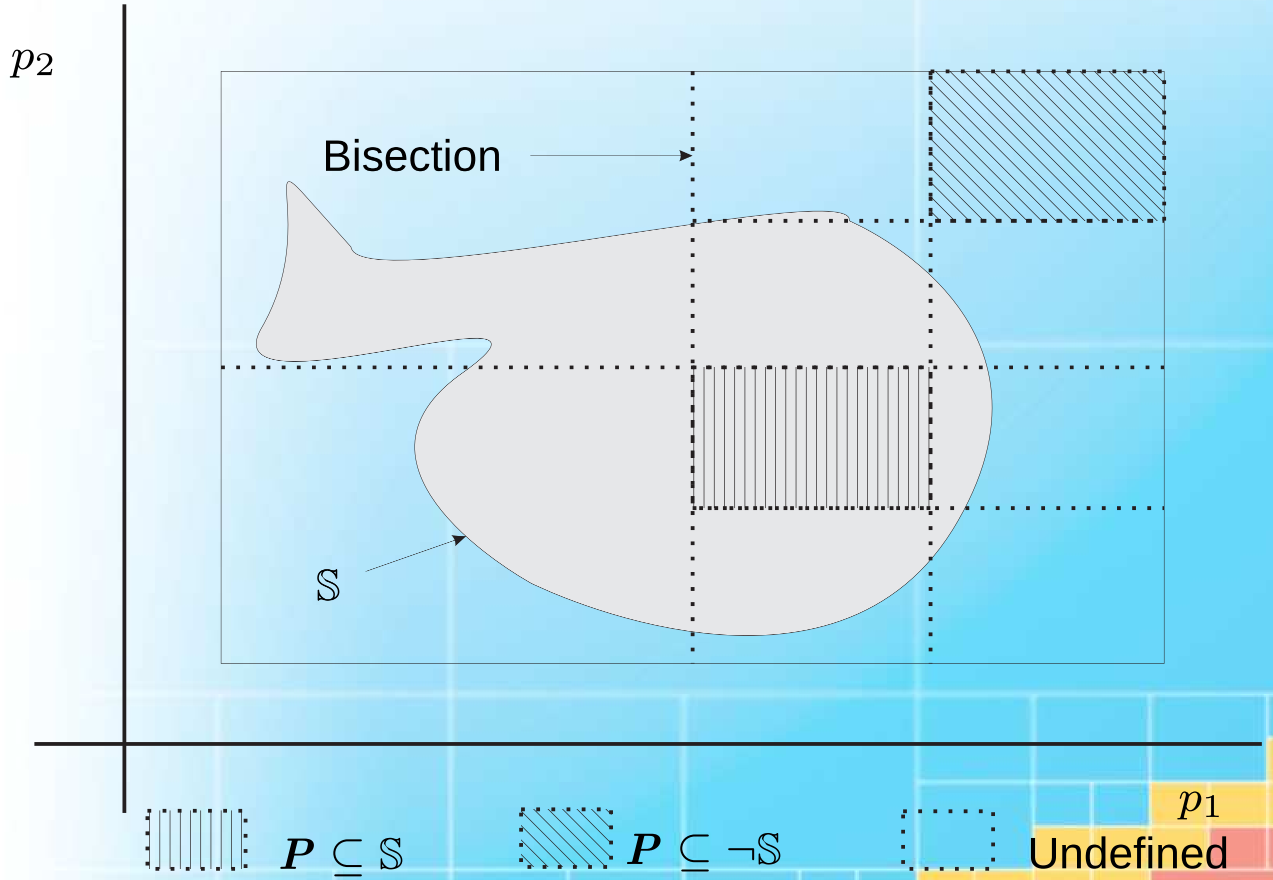
$$\mathbb{S} = \{(\theta, v) | (\exists \delta_s \in [-\frac{\pi}{2}, \frac{\pi}{2}]) \ \ f(\theta, v, \delta_s) = 0\},$$

where

$$f(\theta, v, \delta_s) = \left( (\alpha_r + 2\alpha_f)\, v - 2\alpha_s v_w \cos(\theta + \delta_s) \sin \delta_s + 2\alpha_s v \sin^2 \delta_s \right)^2 + \left( \tfrac{2\alpha_s}{r_r} (\ell - r_s \cos \delta_s) (v_w \cos(\theta + \delta_s) - v \sin \delta_s) \right)^2 - \alpha_r^2 v^2.$$

# Quantified Set Inversion Algorithm

$$\mathbb{S} \triangleq \{\boldsymbol{p} \in \boldsymbol{P} \mid (\exists \boldsymbol{q} \in \boldsymbol{Q})\mathbf{f}(\boldsymbol{p}, \boldsymbol{q}) = \mathbf{0}\},$$



# Quantified Set Inversion Algorithm

$$\mathbb{S} \triangleq \{\boldsymbol{p} \in \boldsymbol{P} \mid (\exists \boldsymbol{q} \in \boldsymbol{Q})\mathbf{f}(\boldsymbol{p}, \boldsymbol{q}) = \mathbf{0}\},$$

$$\textcolor{red}{Rule\,1}: \ \boldsymbol{P} \subseteq \mathbb{S} \Leftrightarrow (\forall \boldsymbol{p} \in \boldsymbol{P})(\exists \boldsymbol{q} \in \boldsymbol{Q}) f(\mathbf{p}, \mathbf{q}) = 0,$$

$$\textcolor{red}{Rule\,2}: \ \boldsymbol{P} \subseteq \neg\mathbb{S} \Leftrightarrow (\forall \boldsymbol{p} \in \boldsymbol{P})\neg((\exists \boldsymbol{q} \in \boldsymbol{Q}) f(\mathbf{p}, \mathbf{q}) = 0),$$

$$\textcolor{red}{Rule\,3}: \ Undefined \Leftrightarrow \neg(Rule\,1) \wedge \neg(Rule\,2).$$

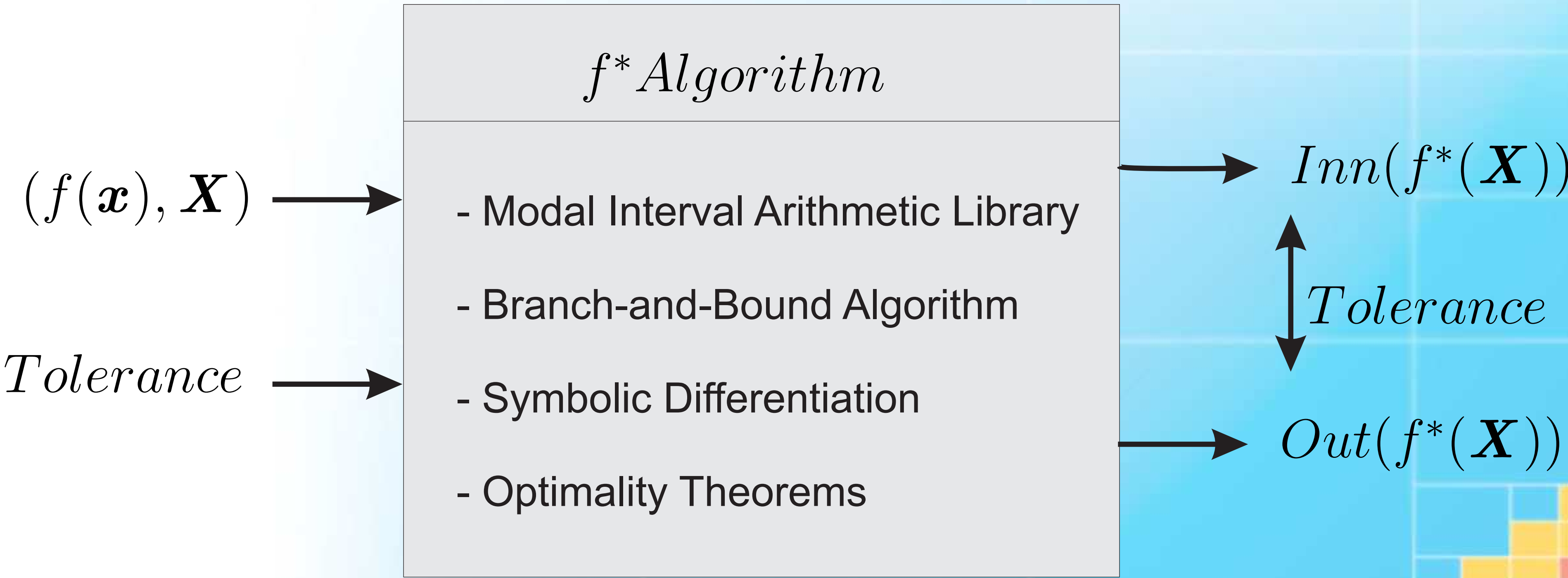
# Modal Interval Analysis

- **MIA** makes equivalent a ∀∃-first order constraint to an interval inclusion.

$$\textit{Rule } 1: \ (\forall \boldsymbol{p} \in \boldsymbol{P})(\exists \boldsymbol{q} \in \boldsymbol{Q}) f(\mathbf{p}, \mathbf{q}) = 0 \Longleftrightarrow f^*(\boldsymbol{P}, \boldsymbol{Q}) \subseteq [0, 0].$$

$$\textit{Rule } 2: \ (\forall \boldsymbol{p} \in \boldsymbol{P})\neg((\exists \boldsymbol{q} \in \boldsymbol{Q}) f(\boldsymbol{p}, \boldsymbol{q}) = 0) \Longleftrightarrow f^*(\boldsymbol{P}, \boldsymbol{Q}) \nsubseteq [0, 0].$$

# Fstar Algorithm



# Fstar Algorithm - Example

- Check if $(\Theta, V) = ([\pi, 2\pi], [2.5, 5]) \subseteq \mathbb{S}$

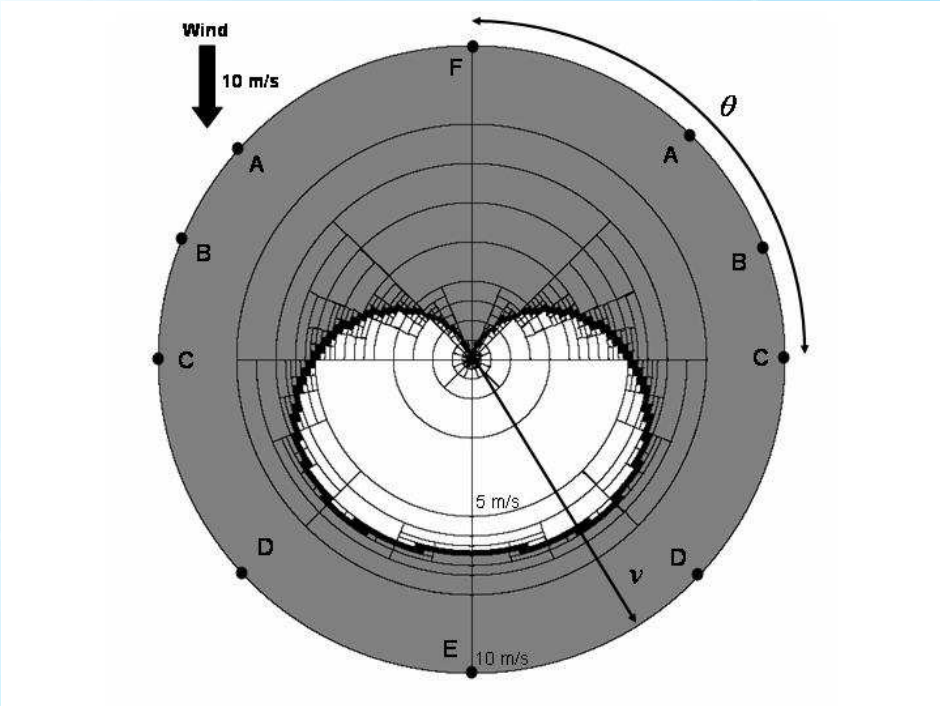
$$
\begin{aligned}
&(\forall\theta \in [\pi, 2\pi])(\forall v \in [2.5, 5])(\exists\delta_s \in [-\pi/2, \pi/2]) \\
&\left((\alpha_r + 2\alpha_f)\, v - 2\alpha_s v_w \cos(\theta + \delta_s) \sin\delta_s + 2\alpha_s v \sin^2\delta_s\right)^2 \\
&+ \left(\frac{2\alpha_s}{r_r} (\ell - r_s \cos\delta_s)\, (v_w \cos(\theta + \delta_s) - v \sin\delta_s)\right)^2 - \alpha_r^2 v^2 \quad = \quad 0.
\end{aligned}
$$

- where $v_w = 10$, $\alpha_f = 60$, $\alpha_r = 300$, $r_s = 1$, $r_r = 2$, $l = 1$.
- Since,

$$Out(fR^*(\theta, v, \delta_s)) = [1.1e+05, -9.3e+03] \subseteq [0, 0] \Rightarrow (\Theta, V) \in \mathbb{S}.$$

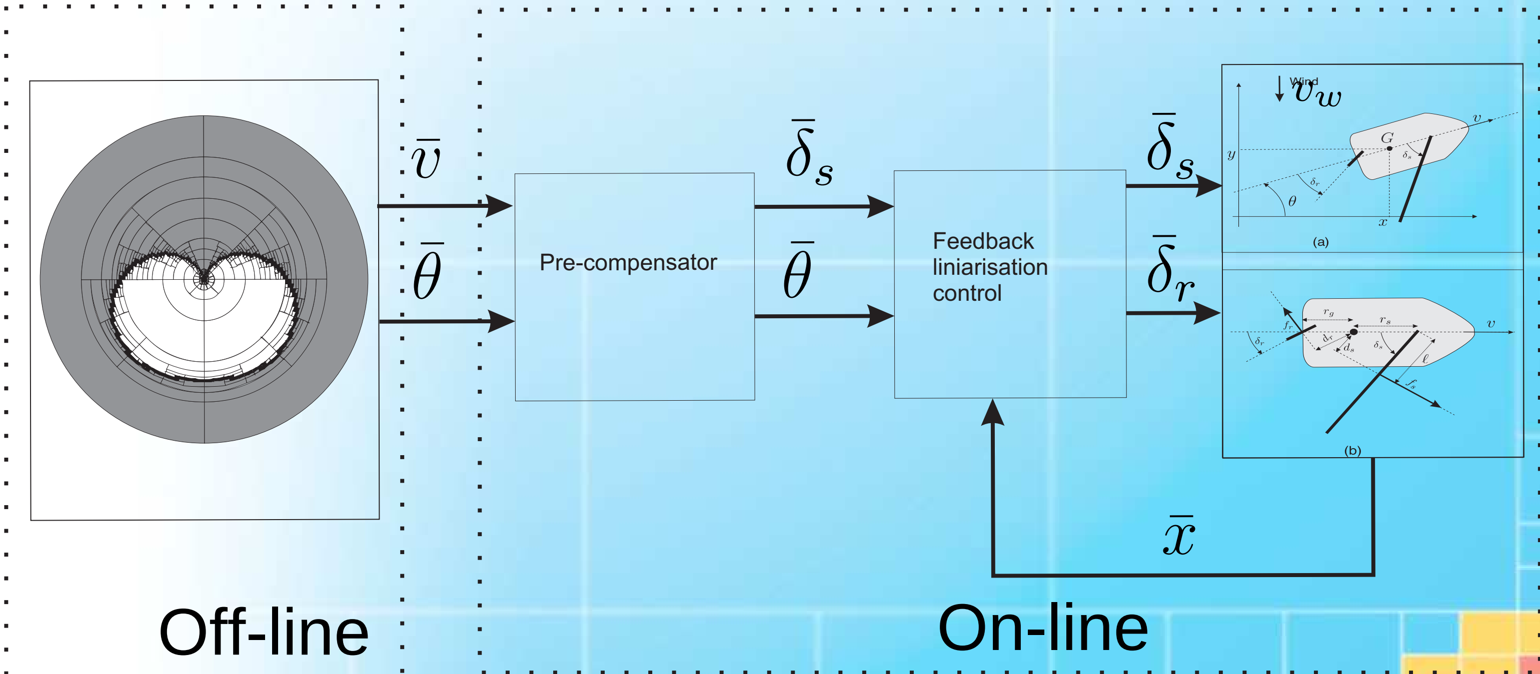
# Polar Diagram - Test Case

- For $v_w = 10$, $\alpha_f = 60$, $\alpha_r = 300$, $r_s = 1$, $r_r = 2$, $l = 1$, and a search space $(\Theta, V)$ of $([0, 2\pi], [0, 20])$, the following solution is obtained.



# Sailboat Control Application

- Objective: Automatically control the speed $v$ and the orientation $\theta$ of a sailboat.



# Control Application