Bidder Optimal Assignments for General Utilities[☆]

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Abstract

We study the problem of matching bidders to items where each bidder i has general, strictly monotonic utility functions $u_{i,j}(p_j)$ expressing his utility of being matched to item j at price p_j. For this setting we prove that a bidder optimal outcome always exists, even when the utility functions are non-linear and non-continuous. We give sufficient conditions under which every mechanism that finds a bidder optimal outcome is incentive compatible. We also give a mechanism that finds a bidder optimal outcome if the conditions for incentive compatibility are satisfied. The running time of this mechanism is exponential in the number of items, but polynomial in the number of bidders.

Keywords: mechanism design, matching markets, discontinuous utilities, envy freeness, lattice

1. Introduction

In matching markets bidders are to be matched to items and the auctioneer receives a monetary compensation from the bidders. Such markets have been studied for several decades (see, e.g., [2]). They have seen a surge of interest with the spread of sponsored search auctions, where advertisers are competing for the assignment of one of a certain number of advertising slots (see, e.g., [3]).

We study the problem of matching bidders to items where each bidder i has general, strictly monotonic utility functions $u_{i,j}(p_j)$ expressing his utility of being matched to item j at price p_j. We do not require that the utility functions are linear, nor do we require that the utility functions are continuous. In addition, we allow every bidder-item pair (i, j) to have a reserve price $r_{i,j}$, i.e., a lower bound on the price that is required if bidder i is matched to item j, and we allow every bidder i to have an outside option o_i, i.e., a lower bound on the utility that bidder i is guaranteed to get even if he is not matched to any item. We can also model per-bidder-item maximum prices $m_{i,j}$, i.e., an upper bound on the price that bidder i is willing to pay for item j, with a discontinuous drop of $u_{i,j}(p_j)$ below o_i at price $p_j = m_{i,j}$. Other settings that can be modeled are interest rates where, e.g., up to a certain price a bidder can still pay from his own pocket but for higher prices he has to borrow money from a bank, leading to a faster drop in utility.

We are interested in outcomes (μ, p) consisting of a matching μ between bidders and items and prices p. An outcome is *feasible* if the price p_j of every matched item j is at least $r_{i,j}$, where i is the bidder that this item is matched to, and if the utility u_i of every bidder i is at least o_i. An outcome is *envy free* if it is feasible and for each bidder i the utility that he gets, i.e., either his outside option or the utility that he gets from being matched to item j at price p_j, is larger than or equal to the utility that he would get if he was matched to any other item k at price p_k. An outcome is *bidder optimal* if it is envy free and if it gives each bidder the highest possible utility among all envy free outcomes. Following earlier work (see, e.g., [4])

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we also consider strategic manipulations by the bidders and analyze *incentive compatible* mechanisms, i.e., mechanisms that ensure that each bidder maximizes his utility by submitting his true utility functions.

We show that a bidder optimal outcome always exists, even when the utility functions are non-linear and non-continuous. We give sufficient conditions under which every mechanism that finds a bidder optimal outcome is incentive compatible. We also give a mechanism that finds a bidder optimal outcome if the conditions for incentive compatibility are satisfied. The running time of this mechanism is exponential in the number of items, but polynomial in the number of bidders. In many applications, including sponsored search, the number of items can be viewed as constant or at least as considerably smaller than the number of bidders. In this case the running time of our mechanism is polynomial in the input size.

2. Related Work

Continuous utility functions. The problem of finding a bidder optimal outcome for *linear* utility functions with *identical* slopes was studied in [5]. They proved the existence of a bidder optimal outcome and showed that it can be found in polynomial time by formulating the problem as a linear program. Later Leonard [6] examined the incentives for misreporting the utility functions and found that the bidder optimal outcome is identical to the outcome of the VCG mechanism [7, 8, 9] and, thus, every mechanism that computes a bidder optimal outcome is incentive compatible. The classic mechanism for linear utility functions with identical slopes is the *Multi-Item Auction* of Demange et al. [10], which is a variant of the *Hungarian Method* by Kuhn [11]. The basic idea of this mechanism is to start with prices all zero and to repeatedly raise the prices of overdemanded items by the same amount. This idea was generalized to *piece-wise linear* utility functions with *non-identical* slopes by Alkan [12, 13], who showed that the prices of overdemanded items need to be raised by different amounts. The resulting mechanism runs in polynomial time and is incentive compatible. The existence of a bidder optimal outcome for more general *non-linear* utility functions was established in [14, 15, 16] and more recently in [17]. Even for these general utility functions every mechanism that computes a bidder optimal outcome is incentive compatible (see, e.g., [16]), but it is not clear whether or under which conditions a bidder optimal outcome can be computed efficiently.

Discontinuous utility functions. In [4] the problem of finding a bidder optimal outcome for *linear* utility functions with *identical* slopes and a *single* discontinuity was studied. They gave a mechanism, which – for inputs in *general position* – finds a bidder optimal outcome in polynomial time and is incentive compatible. Similar results were obtained in [18, 19] and [20]. The problem of finding the smallest envy free prices for a given matching for utility functions of this type was studied in [21]. In [22] a polynomial-time mechanism for *piece-wise linear* utility functions with *non-identical* slopes and *multiple* discontinuities was given. This mechanism – just as the mechanism of [4] – is incentive compatible for inputs in *general position.* For a restricted class of so-called *consistent* utility function an incentive compatible, polynomial-time mechanism was given in [23]. Neither the *piece-wise linear* utility functions studied in [22] nor the *non-linear* utility functions studied here need not be *consistent.*

3. Problem Statement

We are given a set I of n bidders and a set J of k items. We use i to denote a bidder and j to denote an item. For each bidder i and item j we are given a *utility function* $u_{i,j}(p_j)$ expressing bidder i's utility for being matched to item j at price p_j. For each bidder i we are given an *outside option* o_i expressing bidder i's utility for being unmatched. For each bidder i and item j we are given a *reserve price* $r_{i,j}$, i.e., a lower bound on the price p_j that bidder i has to pay if he is matched to item j. We make three assumptions concerning the *input* $(u_{i,j}(\cdot), o_i, r_{i,j})$ consisting of the utility functions $u_{i,j}(\cdot)$, the outside options o_i, and the reserve prices $r_{i,j}$: (i) The utility functions $u_{i,j}(\cdot)$ are strictly monotonically decreasing. (ii) For the outside options o_i there exist *threshold values* $\bar{p}_{i,j}$ such that $u_{i,j}(\bar{p}_{i,j}) \leq o_i$. (iii) The utility functions $u_{i,j}(\cdot)$ need not be globally continuous, but they are locally right-continuous, i.e., $\forall x : \lim_{\epsilon \to 0^+} u_{i,j}(x + \epsilon) = u_{i,j}(x)$.

We are interested in *outcomes* (μ, p) consisting of a *matching* $\mu \subseteq I \times J$ and *prices* p. We require that $p_j \geq 0$ for all $j \in J$. We also require that every bidder is matched to at most one item and that every item is matched to at most one bidder. We do *not* require that all bidders and all items are matched. We use $\mu(i)$ to denote the item that bidder i is matched to and $\mu(j)$ to denote the bidder that item j is matched to. Similarly, we use $\mu(I')$ to denote the set of items matched to bidders in $I' \subseteq I$ and $\mu(J')$ to denote the set of bidders matched to items in $J' \subseteq J$. We use u_i to denote bidder i's utility for outcome (μ, p). The utility of bidder i is $u_i = u_{i,\mu(i)}(p_{\mu(i)})$ if he is matched under μ and it is $u_i = o_i$ if he is not matched under μ.

We say that an outcome (μ, p) is *feasible* if (i) $p_j \geq r_{i,j}$ for all bidder-item pairs $(i, j) \in \mu$ and if (ii) $u_i \geq o_i$ for all bidders $i \in I$. The first condition can be interpreted as individual rationality of the auctioneer and the second as individual rationality of the bidders. We say that an outcome (μ, p) is *envy free* if it is feasible and if $u_i \geq u_{i,j}(p_j)$ for all bidder-item pairs $(i, j) \in I \times J$. In other words, an outcome is envy free if no bidder would get a higher utility if he was matched to a different item. We say that an outcome (μ, p) is *bidder optimal* if it is envy free and if $u_i \geq u'_i$ for all bidders $i \in I$ and every envy free outcome (μ', p'), where u_i denotes bidder i's utility for (μ, p) and u'_i denotes his utility for (μ', p'). A bidder optimal outcome thus gives every bidder the highest possible utility among all envy free outcomes.

Even if a mechanism computes a bidder optimal outcome (μ, p) for all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$, this does not necessarily mean that a bidder cannot benefit from misreporting his utility functions. A mechanism where this is impossible is said to be *incentive compatible*. More specifically, consider an arbitrary bidder i with outside option o_i and utility functions $u_{i,j}(\cdot)$. Then for every two matrices of utility functions u' and u'' with $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$ for i and all j and $u'_{i',j}(\cdot) = u''_{i',j}(\cdot)$ for all $i' \neq i$ and all j and corresponding outcomes (μ', p') and (μ'', p'') of the mechanism we must have $u'_i \geq u''_i$, where u'_i and u''_i denote bidder i's true utility for (μ', p') and (μ'', p''). Note that this definition does *not* involve the $r_{i,j}$ and the o_i. We assume that the $r_{i,j}$ are a property of the seller and cannot be falsified by the bidders. It is also easy to see that misreporting the o_i is never beneficial to i. Overreporting can only lead to a missed chance of being assigned an item and underreporting can lead to a utility below the true outside option.

4. Existence

We begin by proving the existence of a bidder optimal outcome for discontinuous utility functions. The existence of a bidder optimal outcome for continuous utility functions was established in [14, 15, 16] and more recently in [17]. Our proof is similar to the proof in [16] as it establishes the existence of a bidder optimal outcome through a lattice-theoretic argument. It differs from the proof in [16] in that it does *not* require continuity of the utility functions. At the end of this we show that all three conditions (i)–(iii) on the utility functions are required for the existence of a bidder optimal outcome.

Theorem 1. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ there exists a bidder optimal outcome (μ^*, p^*).*

The proof strategy is as follows: The first lemma shows that lowest envy free prices are sufficient for bidder optimality. The second lemma shows that any two envy free outcomes (μ, p) and (μ', p') can be combined to give a new envy free outcome $(\hat{\mu}, \hat{p})$ with utilities $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i and prices $\hat{p}_j = \min(p_j, p'_j)$ for all items j. Thus the set of envy free prices $\{p : \exists\ \mu \text{ s.t. } (\mu, p) \text{ is envy free}\}$ has a unique infimum p^*. Lemma 3 shows that the infimum p^* of this set is in fact a minimum, i.e., there exists a matching μ^* that together with the infimum prices p^* is envy free. Lemma 4 finishes the proof as it shows the existence of at least one envy free outcome.

Lemma 1. *If the outcome (μ^*, p^*) is envy free and if $p^*_j \leq p_j$ for all items j and every envy free outcome (μ, p), then the outcome (μ^*, p^*) is bidder optimal.*

Proof. For a contradiction suppose there exists an envy free outcome (μ', p') with $u'_i > u^*_i$ for at least one bidder i. Since (μ^*, p^*) is feasible this implies that $u'_i > u^*_i \geq o_i$, i.e., bidder i must be matched under μ'. Since (μ^*, p^*) is envy free it follows that $u'_i = u_{i,\mu'(i)}(p'_{\mu'(i)}) > u^*_i \geq u_{i,\mu'(i)}(p^*_{\mu'(i)})$ and, thus, $p'_{\mu'(i)} < p^*_{\mu'(i)}$. This contradicts our assumption that $p^*_j \leq p'_j$ for all items j. □

Lemma 2 (Lattice Lemma)**.** *Any two envy free outcomes (μ, p) and (μ', p') can be combined into an envy free outcome $(\hat{\mu}, \hat{p})$ which has $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i and $\hat{p}_j = \min(p_j, p'_j)$ for all items j.*

Proof. Define $I^- = \{i : u_i > u'_i\}$ and $J^+ = \{j : p_j < p'_j\}$.

First we show that $\mu(I^-) \subseteq J^+$ and that $\mu'(I \setminus I^-) \subseteq J \setminus J^+$. Consider any $(i,j) \in \mu$ with $i \in I^-$. Since $u_i = u_{i,j}(p_j) > u'_i \geq u_{i,j}(p'_j)$ it follows that $p_j < p'_j$ and, thus, $j \in J^+$. We conclude that $\mu(I^-) \subseteq J^+$. Consider any $(i,j) \in \mu'$ with $j \in J^+$. Since $u'_i = u_{i,j}(p'_j) < u_{i,j}(p_j) \leq u_i$ it follows that $u'_i < u_i$ and, thus, $i \in I^-$. We conclude that $\mu'(J^+) \subseteq I^-$ or, equivalently, $\mu'(I \setminus I^-) \subseteq J \setminus J^+$.

The matching $\hat{\mu}$ is identical to μ on $I^- \times J^+$ and identical to μ' on $I \setminus I^- \times J \setminus J^+$. This is a valid matching because – as we have just shown – $\mu(I^-) \subseteq J^+$ and $\mu'(I \setminus I^-) \subseteq J \setminus J^+$, i.e., $\mu(I^-) \cap \mu'(I \setminus I^-) = \emptyset$. The prices $\hat{p}_j$ are identical to p_j for all items $j \in J^+$ and identical to p'_j for all items $j \in J \setminus J^+$. Since $p_j < p'_j$ for all items $j \in J^+$ and $p'_j \leq p_j$ for all items $j \in J \setminus J^+$ we have that $\hat{p}_j = \min(p_j, p'_j)$ for all items j.

Next we show that $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i. If $i \in I^-$ then $u_i > u'_i \geq o_i$ shows that i is matched to some item j under μ. Since $\mu(I^-) \subseteq J^+$ we know that $j \in J^+$, i.e., $\hat{p}_j = \min(p_j, p'_j) = p_j$. It follows that $\hat{u}_i = u_{i,j}(\hat{p}_j) = u_{i,j}(p_j) = u_i > u'_i$. If $i \in I \setminus I^-$, then i is either unmatched or matched to some item j under μ'. In the former case, $i \in I \setminus I^-$ implies that $\hat{u}_i = o_i = u'_i \geq u_i$. In the latter case, $\mu'(I \setminus I^-) \subseteq J \setminus J^+$ implies that $j \in J \setminus J^+$ and, thus, $p'_j \leq p_j$. It follows that $\hat{u}_i = u_{i,j}(\hat{p}_j) = u_{i,j}(p'_j) = u'_i \geq u_i$.

The outcome $(\hat{\mu}, \hat{p})$ is feasible because (i) $\hat{p}_j = \min(p_j, p'_j) = p_j \geq r_{i,j}$ for all bidder-item pairs $(i,j) \in \hat{\mu} \cap (I^- \times J^+)$ and $\hat{p}_j = \min(p_j, p'_j) = p'_j \geq r_{i,j}$ for all bidder-item pairs $(i,j) \in \hat{\mu} \cap ((I \setminus I^-) \times (J \setminus J^+))$ and (ii) $\hat{u}_i = \max(u_i, u'_i) \geq o_i$ for all bidders i. It is envy free because (1) $\hat{u}_i = \max(u_i, u'_i) \geq u_i \geq u_{i,j}(p_j) = u_{i,j}(\hat{p}_j)$ for all bidders $i \in I$ and items $j \in J+$ and (2) $\hat{u}_i = \max(u_i, u'_i) \geq u'_i \geq u_{i,j}(p'_j) = u_{i,j}(\hat{p}_j)$ for all bidders $i \in I$ and items $j \in J \setminus J^+$. □

Lemma 3. *If the set of envy free prices $\{p : \exists\ \mu$ s.t. (μ, p) is envy free$\}$ has a unique infimum p^*, then there must be a matching μ^* that together with p^* is envy free.*

The proof of this lemma in [16] (their Property 2) requires continuity of the functions $u_{i,j}(\cdot)$ to deduce that the set of envy free prices is a *closed* set. Closure then implies that the infimum itself is also contained in the set. As we drop the continuity requirement, this line of argument can no longer be used.

Our proof uses the following definitions: Let $I_> = \{i \in I : \max_j u_{i,j}(p_j) > o_i\}$ denote the set of bidders that get a strictly higher utility from being matched to one of their *first choice items* than from their outside option. The *first choice graph* $G_p = (I_> \cup J, F_p)$ at prices p has a node for every bidder $i \in I_>$, a node for every item $j \in J$, and there is an edge between $i \in I_>$ and $j \in J$ if $j \in \operatorname{argmax}_{j'} u_{i,j'}(p_{j'})$. The *feasible first choice graph* $\tilde{G}_p = (I_> \cup J, \tilde{F}_p)$ at prices p has a node for every bidder $i \in I_>$, and a node for every item $j \in J$, and there is an edge between $i \in I_>$ and $j \in J$ if $j \in \operatorname{argmax}_{j'} u_{i,j'}(p_{j'})$ *and* $p_j \geq r_{i,j}$. For bidder $i \in I$ and item $j \in J$ we define $F_p(i) = \{j : \exists (i,j) \in F_p\}$ and $F_p(j) = \{i : \exists (i,j) \in F_p\}$. For sets of bidders $T \subseteq I$ and sets of items $S \subseteq J$ we define $F_p(T) = \cup_{i \in T} F_p(i)$ and $F_p(S) = \cup_{j \in S} F_p(j)$. We define $\tilde{F}_p(i)$, $\tilde{F}_p(j)$, $\tilde{F}_p(T)$, and $\tilde{F}_p(S)$ analogously.

We call a (possibly empty) set of items $S \subseteq J$ *strictly overdemanded* for prices p with respect to the set of bidders $T \subseteq I$ if (i) $\tilde{F}_p(T) \subseteq S$ and (ii) $\forall\ R \subseteq S, R \neq \emptyset : |\tilde{F}_p(R) \cap T| > |R|$. Using Hall's Theorem [24] one can show that there exists an envy free outcome (μ, p) if and only if no (possibly empty) set of items S is strictly overdemanded at prices p with respect to some set of bidders T.

Proof of Lemma 3. For a contradiction suppose that for the infimum prices p_* there exists no matching μ^* such that (μ^*, p^*) is envy free. Then, by Hall's Theorem [24], there must be a set of bidders T such that the set of items $\tilde{F}_{p^*}(T)$ is strictly overdemanded for prices p^* with respect to T.

In *any* envy free outcome $(\hat{\mu}, \hat{p})$ we have $\hat{p}_j \geq p^*_j$ for all items $j \in J$ and, thus, the strict overdemand for the items in $\tilde{F}_{p^*}(T)$ can only be resolved if (i) at least one of the bidders $i \in T$ has a feasible first choice item $j \in J \setminus F_{p^*}(T)$ under $\hat{p}$ or (ii) for some item $j \in F_{p^*}(T) \setminus \tilde{F}_{p^*}(T)$ we have that $\hat{p}_j \geq r_{i,j}$. Case (i) corresponds, for each pair $(i,j) \in T \times J \setminus F_{p^*}(T)$, to a price increase relative to p^* of $s^i_j = \inf\{x \geq 0 : u_{i,j}(p^*_j + x) \leq \max_{j' \in J \setminus F_{p^*}(T)} u_{i,j'}(p^*_{j'})\}$, which is strictly larger than zero and contained in the set itself as $u_{i,j}(\cdot)$ is right-continuous.[1] Case (ii) corresponds, for each pair $(i,j) \in I \times F_{p^*}(T) \setminus \tilde{F}_{p^*}(T)$, to a price

[1] This no longer holds if the requirement of right-continuity is dropped.

increase relative to p^* of $f^i_j = r_{i,j} - p^*_j$, which is also strictly larger than zero. Let $\delta^i_j = f^i_j$ if $j \in F_{p^*}(i) \setminus \tilde{F}_{p^*}(i)$ and let $\delta^i_j = s^i_j$ otherwise. It follows that $\delta = \min_{i \in I, j \in J \setminus \tilde{F}_{p^*}(T)} \delta^i_j > 0$. Moreover, $\delta > 0$ is a *lower bound* on the sum of the price increases for *any* envy free outcome $(\hat{\mu}, \hat{p})$.

Lemma 2, however, shows that for any $\epsilon > 0$ there exist envy free prices p' such that $|p'_j - p^*_j| < \epsilon$ for all items j. For $\epsilon = \delta/|J|$ this gives a contradiction to the fact that the price increases corresponding to δ were required by *any* envy free outcome. We conclude that for the infimum prices p^* there exists at least one matching μ^* such that (μ^*, p^*) is envy free. □

Lemma 4. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ there exists at least one envy free outcome (μ, p).*

Proof. Let $\mu = \emptyset$ and let $p_j = \max_i(\bar{p}_{i,j})$ for all items $j \in J$, where the $\bar{p}_{i,j}$'s are the threshold values defined above. This outcome is feasible since no item is matched and $u_i = o_i$ for all bidders $i \in I$. It is envy free because $u_i = o_i \geq u_{i,j}(\bar{p}_j) \geq u_{i,j}(p_j)$ for all bidders $i \in I$ and items $j \in J$. □

Proof of Theorem 2. From Lemma 4 we know that there exists at least one envy free outcome (μ, p). By Lemma 2 and Lemma 3 this implies the existence of an envy free outcome (μ^*, p^*) such that $p^*_j \leq p_j$ for all items j and every envy free outcome (μ, p). By Lemma 1 this outcome (μ^*, p^*) is bidder optimal. □

We conclude this section by showing that all three conditions (i)–(iii) on the utility functions (see Section 3) are required to guarantee the existence of a bidder optimal outcome.

Condition (i): There are three bidders and three items. The reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are: $u_{1,1}(x) = u_{3,2}(x) = 1 - x$, $u_{1,2}(x) = u_{3,1}(x) = -x$ and $u_{2,1}(x) = u_{2,2}(x) = 2$ if $x \leq 1$ and $u_{2,1}(x) = u_{2,2}(x) = 3 - x$ otherwise. Then one envy free outcome is $\mu = \{(1,1),(2,2)\}$ and $p = (0,1)$ whereas another envy free outcome is $\mu = \{(2,1),(3,2)\}$ and $p = (1,0)$. In neither of the two outcomes can the price for the item with price 0 be lowered any further without upsetting envy freeness. The first outcome is strictly preferred by the first bidder, whereas the second outcome is strictly preferred by the second bidder.

Condition (ii): There are two bidders and one item. Again, the reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are $u_{i,1}(x) = 1/(1+x)$ for all i. Then no matter how large p_1 is, both bidders will still strictly prefer the item over being unmatched.

Condition (iii): There are two bidders and one item. As before, the reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are $u_{i,1}(x) = 2 - x$ if $x \leq 1$ and $u_{i,1}(x) = -x$ otherwise for all i. Then a price of $p_1 \leq 1$ will not be envy free, as both bidders strictly prefer the item over being unmatched. So any envy free price needs to satisfy $p_1 > 1$ and this set no longer contains its infimum. If we change the first condition of the utility function to $x < 1$, ensuring right-continuity, then the price $p_1 = 1$ is envy free, even though the item cannot be assigned to either of the two bidders.

5. Incentive Compatibility

Next we prove that for discontinuous utility functions every mechanism that computes a bidder optimal outcome is incentive compatible for inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy the following conditions: (a) For every item j there exists a per-item reserve price r_j such that $r_{i,j} = r_j$ for all i. (b) For every *restricted problem* with bidders $I' \subseteq I$, items $J' \subseteq J$, and reserve prices $r'_j \geq r_j$ there exists a bidder optimal outcome (μ, p) such that (b1) $p_j = r'_j$ for all items j unmatched under μ and (b2) if $|I'| \leq |J'|$ then $p_j = r'_j$ for at least one item j matched under μ. At the end of this section we show that for discontinuous utility functions all three conditions (a), (b1), and (b2) are required for incentive compatibility. For continuous utility functions condition (a) is sufficient for incentive compatibility [16].

Theorem 2. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) every mechanism that computes a bidder optimal outcome is incentive compatible.*

We prove this theorem as follows: Consider a set of utility functions $u_{i,j}(\cdot)$. Suppose that for the bidders $i \in I^+$ the $u_{i,j}(\cdot)$ are the true utility functions and that these bidders strictly benefit from reporting utility functions $u'_{i,j}(\cdot)$. Let $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$ for all other bidders $i \in I \setminus I^+$. Then – as we show below – the bidder optimal outcome for $u'_{i,j}(\cdot)$ must be feasible for $u_{i,j}(\cdot)$. If $I^+ = I$, then we get a contradiction from Lemma 5 which shows that no outcome that is feasible for $u_{i,j}(\cdot)$ can give all bidders a strictly higher utility. Otherwise, if $I^+ \subset I$, then we get a contradiction from Lemma 6 which shows that there must be a bidder $i \in I \setminus I^+$ which is *not* envy free with respect to $u_{i,j}(\cdot)$ and, thus, with respect to $u'_{i,j}(\cdot)$.

Lemma 5. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) we have that if the outcome (μ^*, p^*) is bidder optimal, then for no feasible outcome (μ', p') we can have $u'_i > u^*_i$ for all i.*

Proof. Since the input satisfies condition (a) there exist per-item reserve prices r_j for all items j such that $r_{i,j} = r_j$ for all bidders i. Since the input satisfies condition (b) there must be a bidder optimal outcome (μ, p) for the original problem such that $p_j = r_j$ for all items j unmatched under μ and, if $|I| \leq |J|$, such that $p_j = r_j$ for at least one item j matched under μ.

For a contradiction assume that there exists a feasible outcome (μ', p') with $u'_i > u^*_i$ for all bidders i. Since $u_i = u^*_i$ for all bidders i, it follows that $u'_i > u^*_i = u_i$ for all bidders i. Since (μ, p) is feasible this implies that $u'_i > u_i \geq o_i$ for all bidders i and, thus, that (i) all bidders i must be matched under μ'.

Consider an arbitrary bidder-item pair $(i, j) \in \mu'$. Since (μ, p) is envy free, it follows that $u_{i,j}(p'_j) = u'_i > u_i \geq u_{i,j}(p_j)$ and, thus, $p_j > p'_j \geq r_j$. Since $p_j = r_j$ for all items j that are unmatched under μ, this shows that item j must be matched under μ. We conclude that (ii) all items that are matched under μ' are matched under μ and (iii) $p'_j < p_j$ for all of these items j.

From (i) and (ii) we deduce that all items are matched under μ and under μ'. This can only be if $|I| \leq |J|$ and, thus, $p_j = r_j$ for at least one of the items matched under μ. Since the same set of items is matched under μ and under μ' this item j must be matched under μ' and, thus, (iii) shows that $p'_j < p_j = r_j$. We get a contradiction to our assumption that the outcome (μ', p') is feasible. □

Lemma 6. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) we have that if the outcome (μ^*, p^*) is bidder optimal, the outcome (μ', p') is feasible, and $I^+ = \{i \in I \mid u'_i > u^*_i\} \neq \emptyset$, then there exists a bidder-item pair $(i, j) \in I \setminus I^+ \times J$ such that $u'_i < u_{i,j}(p'_j)$.*

Proof. Since the input satisfies condition (a) there exist per-item reserve prices r_j for all items j such that $r_{i,j} = r_j$ for all bidders i. Since the input satisfies condition (b) there must be a bidder optimal outcome (μ, p) for the original problem such that $p_j = r_j$ for all items j unmatched under μ and, if $|I| \leq |J|$, such that $p_j = r_j$ for at least one item j matched under μ.

Since $u_i = u^*_i$ for all bidders i, we have $I^+ = \{i \in I \mid u'_i > u_i\} \neq \emptyset$. Let $\mu(I^+)$ respectively $\mu'(I^+)$ denote the set of items matched to bidders in I^+ under μ respectively μ'. From Lemma 5 we know that $I^+ \neq I$.

Case 1: $\mu(I^+) \neq \mu'(I^+)$. There must be an item $j \in \mu'(I^+)$ such that $j \notin \mu(I^+)$. Let $i' \in I^+$ be the bidder that is matched to item j under μ'. Since $i' \in I^+$ and the outcome (μ, p) is envy free we have that $u_{i',j}(p'_j) = u'_{i'} > u_{i'} \geq u_{i',j}(p_j)$ and, thus, $p_j > p'_j$. Since $p_j = r_j$ for all items j that are unmatched under μ, this shows that item j must be matched under μ. Let $i \in I \setminus I^+$ be the bidder that is matched to item j under μ. Since $i \notin I^+$ and $p_j > p'_j$ we must have that $u'_i \leq u_i = u_{i,j}(p_j) < u_{i,j}(p'_j)$.

Case 2: $\mu(I^+) = \mu'(I^+)$. Let $J^+ = \mu(I^+) = \mu'(I^+)$ and let $u^{-1}_{i,j}(u) := \min_{p_j \in [r_j, \infty)}\{u_{i,j}(p_j) \leq u\}$.[2] Consider the following *restricted problem*: The set of bidders is I^+, the set of items is J^+, the utility functions are $u^+_{i,j}(\cdot) = u_{i,j}(\cdot)$ for all $(i, j) \in I^+ \times J^+$, the reserve prices are $r^+_j = \max(r_j, \max_{i \notin I^+}(u^{-1}_{i,j}(u_i))$ for all $j \in J^+$, and the outside options are $o^+_i = o_i$ for all $i \in I^+$. Since the outcome (μ, p) is envy free for the original problem it is also envy free for the restricted problem. It is even bidder optimal because the existence of an envy free outcome (μ'', p'') for the restricted problem in which at least one bidder $i \in I^+$ has a strictly higher utility would imply the existence of an envy free outcome (μ''', p''') for the original problem with this property and therefore contradict the bidder optimality of (μ, p).

[2] Note that the minimum is contained in the set itself as we only consider right-continuous utility functions.

Case 2.1: The outcome (μ', p') is feasible for the restricted problem. From Lemma 5 we know that there must be a bidder $i \in I^+$ such that $u_i' \leq u_i$. We get a contradiction to the definition of I^+.

Case 2.2: The outcome (μ', p') is *not* feasible for the restricted problem. Since the outcome (μ', p') is feasible for the original problem this can only happen if for some item $j \in J^+$ we have that $r_j^+ > p_j' \geq r_j$ and, thus, $r_j^+ = \max_{i \notin I^+}(u_{i,j}^{-1}(u_i), 0)$. Since $r_j^+ = 0$ would imply $p_j' < r_j^+ = 0$ this can only happen if $r_j^+ = u_{i,j}^{-1}(u_i)$ for some bidder $i \in I \setminus I^+$. Since $i \in I \setminus I^+$ it follows that $p_j' < r_j^+ = u_{i,j}^{-1}(u_i) \leq u_{i,j}^{-1}(u_i')$ and, thus, $u_i' < u_{i,j}(p_j')$. This shows the lemma. □

Proof of Theorem 2. For a contradiction suppose that some subset of bidders $I^+ \subseteq I$ strictly benefits from misreporting their utility functions. Denote the original input by $(u_{i,j}(\cdot), r_j, o_i)$ and the falsified one by $(u_{i,j}'(\cdot), r_j, o_i)$. Note that $u_{i,j}'(\cdot) = u_{i,j}(\cdot)$ for all $(i,j) \in I \setminus I^+ \times J$.

Let (μ^*, p^*) and (μ', p') denote the bidder optimal outcome for the original and falsified input. Denote the utility of bidder i for (μ^*, p^*) and (μ', p') with respect to the original input by u_i^* and u_i'. Denote the utility of bidder i for (μ', p') with respect to the falsified input by u_i''. Note that $I^+ = \{i \in I \mid u_i' > u_i^*\}$.

The outcome (μ', p') is feasible for the original input because (i) $p_j' \geq r_j$ for all items j that are matched under μ' and (ii) $u_i' > u_i^* \geq o_i$ for the bidders $i \in I^+$ and $u_i' = u_i'' \geq o_i$ for the bidders $i \in I \setminus I^+$.

Case 1: $I^+ = I$. Lemma 5 shows that no feasible outcome (μ', p') can give all bidders a strictly higher utility than the bidder optimal outcome (μ^*, p^*). This gives a contradiction.

Case 2: $I^+ \neq I$. Lemma 6 shows that if some feasible outcome (μ', p') gives only some of the bidders a strictly higher utility than the bidder optimal outcome (μ^*, p^*), then there must be a bidder $i \in I \setminus I^+$ and an item $j \in J$ for which $u_i' < u_{i,j}(p_j')$. Since $i \notin I^+$ we have $u_i'' = u_i'$ and $u_{i,j}'(\cdot) = u_{i,j}(\cdot)$. It thus follows that $u_i'' = u_i' < u_{i,j}(p_j') = u_{i,j}'(p_j')$. This contradicts our assumption that the outcome (μ', p') is bidder optimal and therefore envy free for the falsified input. □

We conclude this section with three examples that show that for inputs that violate any of the conditions (a), (b1), or (b2) bidder optimality need not imply incentive compatibility.

Condition (a): There are two bidders and two items. The utility functions are $u_{1,1}(x) = 6 - x$, $u_{1,2}(x) = 5 - x$, $u_{2,1}(x) = 6 - x$, and $u_{2,2}(x) = 6 - x$. The reserve prices are $r_{1,1} = 2$, $r_{1,2} = 0$, $r_{2,1} = 1$, and $r_{2,2} = 2$. The outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \{(1,1),(2,2)\}$ and $p = (2,2)$. If the second bidder reports $u_{2,2}(x) = 0 - x$, then the bidder optimal outcome is $\mu = \{(1,2),(2,1)\}$ and $p = (1,0)$. The utility of the second bidder improves from 4 to 5.

Condition (b1): There are two bidders and two items. The utility functions for $i \in \{1,2\}$ are $u_{i,1}(x) = 10 - x$ for $x < 5$, $u_{i,1}(x) = -\infty$ for $x \geq 5$, $u_{i,2}(x) = 1 - x$ for $x < 1$, and $u_{i,2}(x) = -\infty$ for $x \geq 1$. The reserve prices are $r_1 = r_2 = 0$ and the outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \emptyset$ with $p = (5,1)$. If the second bidder reports $u_{2,1}(x) = -\infty$ for $x \geq 0$, then the bidder optimal outcome is $\mu = \{(1,1),(2,2)\}$ and $p = (0,0)$. The utility of the second bidder improves from 0 to 1.

Condition (b2): There are three bidders and three items. The utility functions are: $u_{1,1}(x) = 6 - x$ and $u_{1,2}(x) = 5 - x$ for $x < 6$ and $u_{1,1}(x) = u_{1,2}(x) = -\infty$ otherwise, $u_{2,1}(x) = 11 - x$ and $u_{2,2}(x) = 5 - x$ and $u_{2,3}(x) = 4 - x$ for $x < 4$ and $u_{2,1}(x) = u_{2,2}(x) = u_{2,3}(x) = -\infty$ otherwise, $u_{3,2}(x) = 10 - x$ and $u_{3,3}(x) = 4 - x$ for $x < 3$ and $u_{3,2}(x) = u_{3,3}(x) = -\infty$ otherwise. The reserve prices are $r_1 = r_2 = r_3 = 0$. The outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \{(1,1),(2,2),(3,3)\}$ and $p = (4,3,2)$. If the second bidder reports $u_{2,1}(x) = -\infty$ for $x \geq 0$, then the bidder optimal outcome is $\mu = \{(1,1),(2,3),(3,2)\}$ and $p = (0,1,0)$. The utility of the second bidder improves from 2 to 4.

6. Mechanism

We conclude with a mechanism that computes a bidder optimal outcome for inputs that satisfy conditions (a) and (b) from the previous section. The mechanism is conceptually simple as it takes a brute force approach by trying all possible matchings μ and all possible *orderings* o of matched bidder-item pairs $(i,j) \in \mu$. For each matching-ordering pair (μ, o) it keeps lower bounds b_j on the prices p_j that it initializes with r_j. It sets the price p_j of all unmatched items j to $b_j = r_j$. For all unmatched bidders it checks (and aborts) if they envy an unmatched item and it makes sure that they do not envy a matched item by updating the lower bounds b_j of all matched items j. It then considers all matched bidder-item pairs $(i, j) \in \mu$ in the order of o and sets $p_j = b_j$, checks (and aborts) if bidder i envies an unmatched item or a previously considered matched item, and updates the bounds of all matched items still to come so that bidder i does not experience any envy. For all matching-ordering pairs (μ, o) for which it does not abort it thus computes a *candidate outcome* (μ, p). After all matching-ordering pairs (μ, o) have been considered it outputs the candidate outcome that gives each bidder the highest possible utility among all candidate outcomes.

Bidder Optimal Outcome
input: utility functions $u_{i,j}(\cdot)$, reserve prices r_j, outside options o_i
output: bidder optimal outcome (μ^*, p^*)

```
μ* = ∅ and p*_j = ∞ for all j
for all matchings μ do
  for all possible orderings o of μ do
    for all items j do
      b_j = r_j
    end for
    for all items j unmatched under μ do
      p_j = b_j
    end for
    for all bidders i unmatched under μ do
      /* check whether bidder i envies an unmatched item j */
      for all items j unmatched under μ do
        if o_i < u_{i,j}(p_j) then
          try next o
        end if
      end for
      /* make sure that bidder i does not envy a matched item j */
      for all items j matched under μ do
        b_j = max(b_j, u_{i,j}^{-1}(o_i))
      end for
    end for
    for all (i, j) ∈ μ in the order of o do
      p_j = b_j
      /* check whether it is feasible to match bidder i to item j */
      if u_{i,j}(p_j) < o_i then
        try next o
      end if
      /* check whether bidder i envies an unmatched item t */
      for all items t unmatched under μ do
        if u_{i,j}(p_j) < u_{i,t}(p_t) then
          try next o
        end if
      end for
      /* check whether bidder i envies a previously considered matched item t */
      for all (s, t) <_o (i, j) do
        if u_{i,j}(p_j) < u_{i,t}(p_t) then
          try next o
        end if
      end for
      /* make sure that bidder i does not envy a not yet considered matched item t */
      for all (s, t) >_o (i, j) do
```

$b_t = \max(b_t, u_{i,t}^{-1}(u_{i,j}(p_j)))$
end for
end for
end for
if $u_i \geq u_i^*$ for all i **then**
$\mu^* = \mu$ and $p_j^* = p_j$ for all j
end if
end for
output (μ^*, p^*)

Theorem 3. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) the mechanism finds a bidder optimal outcome (μ^*, p^*) in time $O((n+k)^k \cdot k^{2k+1} \cdot n)$.*

We proceed as follows: We first show that for all inputs that satisfy conditions (a) and (b) all bidder optimal outcomes (μ^*, p^*) satisfy a certain structural property and that this structural property induces an ordering o^* on the matched bidder-item pairs $(i,j) \in \mu^*$. We then use this to show that for all inputs that satisfy conditions (a) and (b) the mechanism computes *at least one* candidate outcome (μ, p) that is bidder optimal. Afterwards we show that for all inputs that satisfy conditions (a) and (b) *all* candidate outcomes (μ, p) computed by the mechanism are envy free.

Lemma 7. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b), every bidder optimal outcome (μ^*, p^*), and every subset $I^* \times J^* \subseteq \mu^*$ there exists an item $j \in J^*$ for which $p_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$.*

Proof. Consider the *restricted problem* with bidders I^*, items J^*, utility functions $u_{i,j}^*(\cdot) = u_{i,j}(\cdot)$, reserve prices $r_j^* = \max(r_j, \max_{i \notin I^*}(u_{i,j}^{-1}(u_i^*))$, and outside options $o_i^* = o_i$. Since the outcome (μ^*, p^*) is envy free for the original problem its restriction to $I^* \times J^*$ is envy free for the restricted problem. It is even bidder optimal because every envy free outcome (μ', p') for the restricted problem induces an envy free outcome (μ'', p'') for the original problem with $u_i'' = u_i^*$ for all $i \notin I^*$ and $u_i'' = u_i'$ for all $i \in I^*$. Hence for every outcome (μ', p') that is envy free for the restricted problem we must have $u_i' \leq u_i^*$ for all bidders $i \in I^*$.

Since the input $(u_{i,j}(\cdot), r_{i,j}, o_i)$ satisfies conditions (a) and (b) there exists a bidder optimal outcome (μ', p') for the restricted problem in which $p_j' = r_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$ for at least one item j that is matched under μ'. We claim that this item j is matched under μ^* and that $p_j^* = p_j'$.

Item j is matched under μ^* because $j \in J^*$ and *all* items in J^* are matched under μ^*. To see that $p_j^* = p_j'$ assume by contradiction that $p_j^* \neq p_j'$. If $p_j^* < p_j'$ then $p_j^* < p_j' = r_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$ and we either have $p_j^* < r_j$ or $p_j^* = u_{i,j}^{-1}(u_i^*)$, and thus, $u_i^* < u_{i,j}(p_j^*)$ for some bidder $i \notin I^*$. In the former case we get a contradiction to the fact that (μ^*, p^*) is feasible and in the latter case we get a contradiction to the fact that (μ^*, p^*) is envy free. If $p_j^* > p_j'$ then for the bidder i that is matched to item j under μ^* we have $u_i^* = u_{i,j}(p_j^*) < u_{i,j}(p_j') \leq u_i'$ because $p_j^* > p_j'$ and (μ', p') is envy free. This contradicts the fact that since (μ^*, p^*) is bidder optimal and (μ', p') is envy free for the restricted problem we must have $u_i^* \geq u_i'$. □

By the previous lemma for all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) every bidder optimal outcome (μ^*, p^*) induces an ordering o^* on the bidder-item pairs $(i,j) \in \mu^*$ as follows:

Induced Ordering
input: bidder optimal outcome (μ^*, p^*)
output: ordering o^*
$I^* \times J^* = \mu^*$, $o^* = \emptyset$
while $I^* \times J^* \neq \emptyset$ **do**
add $(i,j) \in I^* \times J^*$ for which $p_j^* = \max(r_j, \max_{s \notin I^*} u_{s,j}^{-1}(u_s^*))$ to o^*
remove (i,j) from $I^* \times J^*$
end while
output o^*

Lemma 8. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) the mechanism computes at least one candidate outcome (μ, p) that is bidder optimal.*

Proof. Since the input $(u_{i,j}(\cdot), r_{i,j}, o_i)$ satisfies conditions (a) and (b) there exists a bidder optimal outcome (μ^*, p^*) such that $p^*_j = r_j$ for all items j unmatched under μ^*. Let $I^*_{\not\in\mu^*}$ and $J^*_{\not\in\mu^*}$ denote the bidders and items that are unmatched under μ^*. For every ordering o^* and all m with $0 \leq m \leq |\mu^*|$ let I^*_m and J^*_m denote the first m bidders and items that are matched under μ^*. By Lemma 7 the bidder optimal outcome (μ^*, p^*) induces an ordering o^* such that for all m with $0 \leq m \leq |\mu^*|$ we have $p^*_j = \max(r_j, \max_{s \in I^*_{\not\in\mu^*} \cup I^*_{m-1}} u^{-1}_{s,j}(u^*_s))$, where j is the m-th item matched under μ^*.

We claim that when the mechanism considers μ^* and o^*, then it computes a candidate outcome (μ, p) with $\mu = \mu^*$ and $p = p^*$. It suffices to show that for all m with $0 \leq m \leq |\mu^*|$ we have (1) $p_j = p^*_j$ for all items $j \in J^*_{\not\in\mu^*} \cup J^*_m$ and (2) $u_i = u^*_i$ for all bidders $i \in I^*_{\not\in\mu^*} \cup I^*_m$.

For $m = 0$ we have $J^*_0 = \emptyset$ and $I^*_0 = \emptyset$. Hence the items in $J^*_{\not\in\mu^*} \cup J^*_0$ and the bidders in $I^*_{\not\in\mu^*} \cup I^*_0$ are precisely the unmatched items $J^*_{\not\in\mu^*}$ and bidders $I^*_{\not\in\mu^*}$. For every unmatched item $j \in J^*_{\not\in\mu^*}$ we have $p_j = b_j = r_j = p^*_j$ (lines 4–9) and for every unmatched bidder $i \in I^*_{\not\in\mu^*}$ we have $u_i = o_i = u^*_i$.

For $m > 0$ assume that for all s with $0 \leq s \leq m-1$ we have (1) $p_j = p^*_j$ for all items $j \in J^*_{\not\in\mu^*} \cup J^*_s$ and (2) $u_i = u^*_i$ for all bidders $i \in I^*_{\not\in\mu^*} \cup I^*_s$. For the m-th matched item j we have $p_j = b_j = \max(r_j, \max_{s \in I^*_{\not\in\mu^*} \cup I^*_{m-1}} u^{-1}_{s,j}(u_s))$ (lines 4–6, 16–18, and 35–37). Since by induction $u_s = u^*_s$ for all $s \in I^*_{\not\in\mu^*} \cup I^*_{m-1}$ this shows that $p_j = \max(r_j, \max_{s \in I^*_{\not\in\mu^*} \cup I^*_{m-1}} u^{-1}_{s,j}(u^*_s)) = p^*_j$. For the m-th matched bidder i we thus have $u_i = u_{i,j}(p_j) = u_{i,j}(p^*_j) = u^*_i$. □

Lemma 9. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) all candidate outcomes (μ, p) computed by the mechanism are envy free.*

Proof. Feasibility follows from the fact that $p_j \geq r_j$ for all items j (lines 4–6), that the utility u_i of all unmatched bidders i is o_i by definition, and that for all matched bidders i with $(i, j) \in \mu$ we have $u_i = u_{i,j}(p_j) \geq o_i$ (lines 22–24). Envy freeness follows from the fact that for all unmatched bidders i we have $u_i = o_i \geq u_{i,j}(p_j)$ for all items j (lines 12–14 and lines 16–18) and that for all matched bidders i with $(i, j) \in \mu$ we have $u_i = u_{i,j}(p_j) \geq u_{i,t}(p_t)$ for all items t (lines 26–28, lines 31–33, and lines 35–37). □

Proof of Theorem 3. For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) the mechanism outputs a bidder optimal outcome (μ^*, p^*) because it computes at least one candidate outcome (μ, p) that is bidder optimal by Lemma 8 and all candidate outcomes (μ, p) are envy free by Lemma 9.

For the running time observe that there are $O((n+k)^k \cdot k^k)$ different matchings of k items to n bidders as there are $\binom{n+k}{k} = O((n+k)^k)$ ways to choose the sets of items and bidders. Observe further that there are up to $k! = O(k^k)$ matchings for a particular choice and up to $k! = O(k^k)$ possible ways of ordering the up to k matched bidder-item pairs. Finally, observe that checking a particular matching-ordering pair takes time $O(nk)$. Hence the total running time is $O((n+k)^k \cdot k^{2k+1} \cdot n)$. □

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